1. Solve the following differential equations. (40%)

   (a) $(2x^3 - xy^2 - 2y + 3)dx - (x^2y + 2x)dy = 0$

   (b) $xy' = 2y + x^3 e^x$

   (c) $y'' + 2y' + y = xe^{-x}$

   (d) $4y'' + \pi^2 y = 0,\quad y(0) = 2, y'(0) = 0$

2. Consider the second-order homogeneous linear differential equation,
$$y'' + p(x)y' + q(x)y = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(*)$$
   (a) Let y_1 and y_2 be two solutions of Eq (*), prove that $c_1 y_1 + c_2 y_2$ is also a solution of Eq (*). (4%)

   (b) Suppose $y_1 = \sin^2 x$ and $y_2 = \cos^2 x$, can they form a basis of Eq (*)? Why? (3%)

   (c) Now consider the second-order non-homogeneous linear differential equation
$$y'' + p(x)y' + q(x)y = r(x), r(x) \neq 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(**)$$
   and let y_1 and y_2 be two solutions of Eq (**), is $c_1 y_1 + c_2 y_2$ also a solution of Eq (*)? (3%)

3. Solve the following system of differential equations (15%)
$$y_1' = -y_1 + y_2 \,,\; y_2' = -y_1 - y_2 \;;\; y_1(0) = 0 \,,\; y_2(0) = 1$$

4. Solve the initial value problem (15%)
$$y'' + 3y' + 2y = f(t) \,;\; y(0) = 0 \,,\; y'(0) = 1$$
   in which $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & otherwise \end{cases}$

5. Determine the Laplace transform of

   (a) $f(t) = 1$. (5%)

   (b) $f(t) = \dfrac{e^{-2t}\sin t}{t}$ (5%)

6. Determine the Laplace inverse transform of

   (a) $F(s) = 1$. (5%)

   (b) $\dfrac{e^{-5s}}{s(s+1)^2}$ (5%