國立中央大學九十一學年度轉學生入學試題卷

數學系 三年級　　科目：高等微積分　　共 1 頁 第 1 頁

1. (20%) Evaluate the following limits if they exist:

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|xy|^{3/2}}{x^2+y^2}$,　　(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x \sin y}{x^2+y^2}$.

2. (20%) Let

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

   Compute $f_x(0,0)$, $f_y(0,0)$, $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

3. (20%) Let f be continuous on the bounded closed interval $[a,b]$, and $f(x) \neq 0$ on $[a,b]$.

   (a) Show that there exists $M > 0$ such that $|f(x)| \geq M$ for all $x \in [a,b]$.

   (b) Let $\{f_n\}$ be a sequence of functions such that $f_n \to f$ uniformly on $[a,b]$. Prove that $\dfrac{1}{f_n}$ is defined for large n, and $\dfrac{1}{f_n} \to \dfrac{1}{f}$ uniformly on $[a,b]$ as $n \to \infty$.

4. (20%) Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Let $E \subset \mathbb{R}^n$.

   (a) Give the definition that $f : E \to \mathbb{R}^m$ be uniformly continuous on E. What is the difference between "continuous on E" and "uniformly continuous on E"?

   (b) Prove that if $f : E \to \mathbb{R}^m$ is continuous on E and if E is a compact set, then f is uniformly continuous on E.

5. (20%) Let C be the unit circle $x^2 + y^2 = 1$ traversed counterclockwise.

   (a) Evaluate the line integral

$$\int_C \frac{y}{x^2+y^2}\,dx - \frac{x}{x^2+y^2}\,dy.$$

   (b) Let $F(x,y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2}\right)$. Is $F(x,y)$ a gradient? Why?

參考