參考用

1. Evaluate

$$\frac{1}{2\pi i}\oint_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz$$

around the circle C with equation |Z|=3. (10%)

2. Evaluate

(a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ (7%)

(b) $\lim\limits_{x\to 0}\dfrac{\sin(\tan x)-\tan(\sin x)}{x^7}$ (7%)

3. Find the (a) eigenvalues and (b) eigenvectors of

$$A=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$ (14%)

4. Show that

$$\nabla^2\vec{u}=grad\ div\vec{u}-rot\ rot\vec{u}$$

where $\nabla^2\vec{u}$ is called vector Laplacian and defined by
$\nabla^2\vec{u}=(\nabla^2u_1,\nabla^2u_2,\nabla^2u_3)$ *as* $\vec{u}=(u_1,u_2,u_3)$.
$rot\vec{u}$ is also written as $curl\vec{u}$. (10%)

5. Solve the differential equation

$$y''-3y'+2y=2e^{-t},\ y(0)=2,\ y'(0)=-1.$$

(12%)

參考用

6. Prove that $\Gamma(1/2)=\sqrt{\pi}$, where $\Gamma(x)$ is the Gamma function. (10%)

7. Using the generating function of Bessel function

$$g(x,t)=e^{(x/2)(t-1/t)}=\sum_{n=-\infty}^{\infty} J_n(x)t^n$$

and the orthoganality properties of cosine and sine:

$$\int_0^{\pi}\cos m\theta\cos n\theta=\begin{cases}0 & m\neq n\\ \pi/2 & m=n\end{cases}$$

$$\int_0^{\pi}\sin m\theta\sin n\theta=\begin{cases}0 & m\neq n\\ \pi/2 & m=n\end{cases}$$

to show that $J_n(x)$ can be written as

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos(n\theta-x\sin\theta)d\theta,\ n=0,1,2,\ldots$$

where $J_n(x)$ is the Bessel function.

hint: let $t=e^{i\theta}$ in $g(x,t)$ (15%)

8. Solve the one-dimensional partial differential equation

$$\frac{\partial^2 q(x,\tau)}{\partial x^2}=\frac{\partial q(x,\tau)}{\partial \tau},$$

with the initial condition

$$q(x,0)=S\delta(x),$$

where $\delta(x)$ is the delta function: $\delta(x)=\begin{cases}1 & x=0\\ 0 & \text{elsewhere}\end{cases}$. (15%)