國立中央大學八十八學年度碩士班研究生入學試題卷

所別：大氣物理研究所　不分組　科目：應用數學　共 / 頁 第 / 頁

1. (a) Find the Fourier integral representation of the non-periodic function

$$f(x)=\begin{cases}1 & \text{if } |x|<1,\\ 0 & \text{if } |x|>1.\end{cases}$$ *(10 %)*

(b) Based on (a), show that $\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$. *(5 %)*

2. (a) Solve the heat equation given by

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \qquad (c \text{ is real})$$

for a finite bar at length of 2π with the initial condition $u(x, 0) = f(x)$ where

$$f(x)=\begin{cases}x & \text{if } 0<x<\pi,\\ 2\pi-x & \text{if } \pi<x<2\pi.\end{cases}$$ *(15 %)*

(b) Obtain the steady-state solution of u. *(5 %)*

3. Find the general solution **y** for a coupled first-order system given by $\mathbf{y}'=\mathbf{A}\mathbf{y}+\mathbf{g}e^{2t}$ where

$$\mathbf{y}=\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \text{ and } \mathbf{g}=\begin{bmatrix}6\\ -3\end{bmatrix}.$$ *(15 %)*

4. Solve the second-order differential equation for $y(t)\ (t\ge 0)$ ***using Laplace transform***:

$$y''+\omega^2 y=0, \qquad y(0)=A, y'(0)=B$$

where ω, A and B are real (all not zero). *(15 %)*

5. Transform the quadratic form $Q=17x_1^2-30x_1x_2+17x_2^2=128$ to the principal axes in the y_1y_2-coordinates and geometrically show the rotation angle between the original axes in the x_1x_2-coordinates and the principal axes. *(15 %)*

6. Show that the inverse $\mathbf{A}^{-1}$ (if exists) has the eigenvalues of $1/\lambda_1,\cdots,1/\lambda_n$ where $\lambda_1,\cdots,\lambda_n$ are eigenvalues of the $n\times n$ matrix **A**. *(10 %)*

7. For a cloud in the sky, the local time change rate is given by

$$\frac{\partial \rho}{\partial t}=-\nabla\cdot\rho\mathbf{V}$$

where ρ is the cloud density and **V** is a three-dimensional wind vector.

Show that the total mass of the cloud must be conserved at any time.

(Hint: Apply Gauss divergence theorem) *(10 %)*