國立中央大學八十九學年度碩士班研究生入學試題卷

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參考用

(1) (25 points)

(a) The adjoint $M^\dagger$ of a matrix M is defined as the transpose of the complex conjugate of M, i.e., $M^\dagger = (M^*)^t$. A matrix is called normal if it commutes with its adjoint, i.e., $[M, M^\dagger] = MM^\dagger - M^\dagger M = 0$. Show that the eigenvectors correspond to different eigenvalues of a normal matrix are orthogonal to each other.

(b) Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ \epsilon^2 & 1 \end{pmatrix}.$$

When $\epsilon \to 0$, what happens to the eigenvectors of the above two matrices.

(2) (25 points)

Plane elliptic coordinates (u, v) is defined as $x = a\cosh(u)\cos(v)$, $y = a\sinh(u)\sin(v)$, where (x, y) is the Cartesian coordinates.

(a) If u is a constant, varying v describes a curve on the plane called coordinate curve of u. Find the equation of this curve in terms of x and y. Similarly, work out the coordinate curve of v. Sketch the coordinate curves (for different u and v) in the x-y plane.

(b) Calculate $\partial/\partial x$ and $\partial/\partial y$ in elliptic coordinates.

(3) (25 points)

Define Laplace transform as:

$$L\{f(t)\} \equiv F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt\,.$$

(a) Derive the Laplace transforms of $t^n f(t)$ and $d^m f(t)/dt^m$. Thus find the Lapace transform of $t^n d^m f(t)/dt^m$.

(b) Derive the Laplace transforms of $\cos(t)$ and $\sin(t)$. Hence solve the equation for $t > 0$

$$\frac{dx}{dt} + ax = b\cos(t)\,,$$

where a and b are constants, and $x(0) = 1$.

(4) (25 points)

Consider the linear partial differential equation

$$\frac{\partial^2 U}{\partial t^2} = P(x,t)\frac{\partial^2 U}{\partial x^2}\,,$$

where $P(x,t) > 0$ for all x and t.

(a) What is the meaning of linear? What is the name of this equation? Find the general solution of the equation if $P(x,t) \equiv 1$.

(b) A function $U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$ satisfies the partial differential equation above, and the boundary conditions $U_n(1,t) = U_n(2,t) = 0$. If

$$U_n(x,t) = [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]\sqrt{x}\,\sin[\kappa_n \log_e(x)]\,,$$

where A_n and B_n are constants, find $P(x,t)$ and all possible values of κ_n and ω_n.