國立中央大學八十三學年度研究所碩士班入學試題卷

系所別： 天文研究所　　組　　科目： 應用數學　　共 一 頁 第 一 頁

參考用

PLEASE READ THIS SHORT MESSAGE FIRST:
Please work out the following problems in detail, otherwise put down how you may proceed. Attempt as many problems as you can but spend your time *wisely*. Please pay attention to the score of each problem. Good luck!

(1) (20 points) What is the name of the following equation?

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

What is the meaning of a? If a is a constant, show that $u = g(s)$ is a solution of the above equation, where $g(s)$ is an arbitrary analytic function of s, and s is given by $sx - \sqrt{1-s^2}\,y = at$.

(2) (20 points) What is a linear ordinary differential equation? Give one property of its solutions. Solve the following set of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2x + 3y\,,$$

subject to the conditions $x = x_\circ$, $y = y_\circ$ at $t = 0$.

(3) (20 points) What is divergence theorem in the context of vector analysis? Show that

$$\nabla^2 \frac{1}{|\vec{r} - \vec{r}_\star|} = -4\pi\delta(|\vec{r} - \vec{r}_\star|)\,,$$

where the differentiation is with respect to $\vec{r}$. [Hints: first show that the left hand side is zero if $\vec{r} \neq \vec{r}_\star$. Then use divergence theorem to show that the integral of the left hand side over a volume including $\vec{r}$ is equal to -4π.]
Consider a vector $\vec{V} = -\vec{\nabla}\phi + \vec{\nabla} \times \vec{A}$, where

$$\phi(\vec{r}) = \frac{1}{4\pi}\int \frac{s(\vec{r}_\star)}{|\vec{r} - \vec{r}_\star|}\,\mathrm{d}^3 r_\star\,, \qquad \vec{A}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\omega}(\vec{r}_\star)}{|\vec{r} - \vec{r}_\star|}\,\mathrm{d}^3 r_\star\,.$$

Show that $\vec{\nabla}\cdot\vec{V} = s$ and $\vec{\nabla}\times\vec{V} = \vec{\omega}$. [Hints: use the first result of this problem.]
Suppose $\vec{V} = \vec{\Omega}\times\vec{r}$. If $\vec{\Omega}$ is a constant, find $\vec{\omega}$. Can you find $\vec{A}$? If yes, what is it? If no, why?
[Note: $\vec{\nabla}\times(\vec{P}\times\vec{Q}) = \vec{P}(\vec{\nabla}\cdot\vec{Q}) - \vec{Q}(\vec{\nabla}\cdot\vec{P}) + (\vec{Q}\cdot\vec{\nabla})\vec{P} - (\vec{P}\cdot\vec{\nabla})\vec{Q}$]

(4) (20 points) $f(z) = u(x,y) + i\,v(x,y)$ is an analytic function of $z = x + i\,y$, where u, v, x and y are real and $i = \sqrt{-1}$. Write down the Cauchy-Riemann relation. If $z = r\exp(i\theta)$, write down the Cauchy-Riemann relation in polar coordinates (r,θ). What is $\partial f/\partial \bar{z}$? Show that $\partial f/\partial \bar{z} = 0$ is equivalent to the Cauchy-Riemann relation ($\bar{z} = x - i\,y$ is the complex conjugate of z).
What is a Laurent series? What is the relation between Laurent series and residues? Find the Laurent series of $g(z) = z^{-1}(1-z)^{-1}$ (please state the range of validity of each series). Write down the residue of $g(z)$ at each singular point.

(5) (20 points) What is a Fourier series? Consider the one dimensional inhomogeneous diffusion equation

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + s(t,x)\,,$$

where the source $s(t,x)$ is a given function. This equation is subject to initial condition $u(0,x) = f(x)$ when $t = 0$, and boundary conditions $u(t,0) = u(t,L) = 0$ at $x = 0$ and L. Try the following Fourier series

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{L}\,, \qquad s(t,x) = \sum_{n=1}^{\infty} s_n(t)\sin\frac{n\pi x}{L}\,.$$

Find the ordinary differential equation governing $u_n(t)$. What is the corresponding initial or boundary condition on $u_n(t)$? If $s_n(t) = \exp(-t/\tau_n)$ and $f(x) = \sin(\pi x/L)$, write down the solution $u(t,x)$.