所別：統計研究所碩士班 一般生 科目：數理統計
學位在職生

Note: There could be more than one answer for each problem.

(1) **(10%)** Let X be a random variable with probability density function $f(x) = \beta\alpha^{\beta} / x^{\beta+1}$, $\alpha < x < \infty$, $\alpha, \beta > 0$. Denote the *cdf* of X by $F(x) = P(X \le x)$ and let $U = F(X)$, $V = 1 - F(X)$.
Which of the followings are true?
(a) $F(x)$ is strictly increasing and differentiable for all x, $\alpha < x < \infty$.
(b) U and V have the same probability distribution.
(c) $E[V] = \beta\alpha / (\beta - 1)$.
(d) $-\log(U)$ has an exponential distribution with mean 1.
(e) U / V has a Cauchy distribution.

(2) **(10%)** Which of the followings are true?
(a) The exponential random variable is the only one random variable having the "memoryless" property.
(b) Let X_1 and X_2 be independent exponential random variables, then $\min(X_1, X_2)$ is also an exponential random variable.
(c) Let X_1 and X_2 be independent exponential random variables with mean λ_1 and λ_2 respectively, then $X_1 + X_2$ is an exponential random variable with mean $\lambda_1 + \lambda_2$.
(e) Moments of all orders for the exponential random variables always exist.
(d) If X_1 and X_2 are two arbitrary random variables, then $Cov(X_1, X_2) = 0$ if and only if X_1 and X_2 are independent.

(3) **(15%)** If X and Y are any two random variables, which of the followings are true?
(a) $E[Y] = E[E(X \mid Y)]$
(b) $\mathrm{Var}(X) \ge \mathrm{Var}(E[X \mid Y])$
(c) $[\mathrm{Cov}(X, Y)]^2 \le \mathrm{Var}(X)\mathrm{Var}(Y)$
(d) If X and Y are independent, then $g(X)$ and $h(Y)$ are independent for any functions g and h.
(e) Let X and Y be iid from $N(\theta, 1)$, then $E[(X_1 + X_2)/2 \mid X_1]$ is an unbiased estimator of θ.

(4) **(15%)** Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution, and let $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Which of the followings are true?
(a) $\overline{X}$ and S^2 are not independent.
(b) $(n-1)S^2/\sigma^2$ has a chi squared distribution with n degrees of freedom.
(c) When $n \to \infty$, the distribution of $\dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ converges to a standard normal distribution.
(d) The quantity $n(\overline{X} - \mu)^2 / S^2$ has the F distribution with 1 and $n-1$ degrees of freedom.
(e) The quantity $\dfrac{S^2}{n(\overline{X} - \mu)^2}$ has the F distribution with $n-1$ and 1 degrees of freedom.

注意：背面有試題

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(5) **(15%)** Let $X_1,\ldots,X_n$ be a random sample from a Bernoulli(p) distribution and let $\bar{X}=(1/n)\sum_{i=1}^{n}X_i$. Which of the followings are true?

(a) $\bar{X}$ is a consistent statistic for p.

(b) $\sum_{i=1}^{n}X_i$ is a minimal sufficient statistic for p.

(c) $(\bar{X})^2$ is the maximum likelihood estimator for p^2.

(d) $X_1(1-X_2)$ is an unbiased estimator for $p(1-p)$.

(e) For any $i\neq j$, $X_i(1-X_j)$ is the UMVUE of $p(1-p)$.

(6) **(15%)** Let $X_1,\ldots,X_n$ be a random sample from $N(\mu,1)$, μ is unknown. Consider a likelihood ratio test for $H_0:\mu=\mu_0$ vs $H_1:\mu\neq\mu_0$ that has a rejection region $\{\vec{x}:\lambda(\vec{x})\leq c\}$, $0\leq c\leq 1$. Which of the followings are true?

(a) The statistic $\lambda(\vec{x})$ can be defined as $\dfrac{L(\hat{\mu}\mid\vec{x})}{L(\mu_0\mid\vec{x})}$, where L is the likelihood function and $\hat{\mu}$ is the maximum likelihood estimator of μ.

(b) According to Neyman-Pearson, this likelihood ratio test is also uniformly most powerful.

(c) If the level of this test is α, the rejection region can be written as $\{\vec{x}:\sqrt{n}\,|\,\bar{x}-\mu_0\,|\leq z_{\alpha/2}\}$, where $z_{\alpha/2}$ is such that $P(Z>z_{\alpha/2})=\alpha/2$, Z is a standard normal random variable.

(d) $[\bar{x}-z_{\alpha/3}/\sqrt{n},\bar{x}+z_{2\alpha/3}/\sqrt{n}]$ is a $1-\alpha$ confidence interval for μ.

(e) Any $1-\alpha$ confidence intervals for μ must have the interval length at least $2z_{\alpha/2}/\sqrt{n}$.

(7) **(20%)** Suppose that $X_1,\ldots,X_n$ is a random sample from *Uniform*$(0,\theta)$. In order to make inference about the unknown parameter θ, let's consider a decision rule which rejects some null hypothesis H_0 when $X_{(n)}\geq c$, where $X_{(n)}=\max(X_1,\ldots,X_n)$.

(a) What is the probability density function of $X_{(n)}$?

(b) Compute the power function for this hypothesis.

(c) Determine the rejection region (that is, find the value of c) of a level $\alpha=0.05$ test for $H_0:\theta\leq 1/2$ vs $H_1:\theta>1/2$.

(d) Continue from part (c), can you reduce the Type I Error and Type II Error probabilities at the same time? If yes, what is your strategy?

(e) If the sample size $n=2$, the observed value $X_{(n)}=0.48$, what is the p-value? Based on the p-value, would you reject or accept H_0?