所別：統計研究所碩士班 一般生 科目：基礎數學

1. Let $f(x)$ be a real-valued function defined in the open interval (-1,1). Suppose that $\lim_{h\to 0}\frac{f(h)-f(-h)}{2h}$ exists. Prove or disprove that $f'(0)$ exists. *(10%)*

2. Suppose $a_n > 0$, $n = 1, 2, \cdots$, $s_n = a_1 + \cdots + a_n$, and $\sum_{n=1}^{\infty} a_n$ diverges. Prove that (a) $\sum_{n=1}^{\infty}\frac{a_n}{1+a_n}$ diverges (b) $\sum_{n=1}^{\infty}\frac{a_n}{s_n^2}$ converges. *(5+5=10%)*

3. (a) Use the formula $\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$, to show that $\int_0^{\infty}\frac{\sin x\cos x}{x}dx = \frac{\pi}{4}$. *(5%)*

   (b) Use integration by parts in (a), to find $\int_0^{\infty}\frac{\sin^2 x}{x^2}dx$. *(5%)*

   (c) Use (b) and $\sin^2 x + \cos^2 x = 1$ to find $\int_0^{\infty}\frac{\sin^4 x}{x^2}dx$. *(8%)*

4. Let $f_n(x) = 1/(1+n^2x^2)$, $n = 1, 2, \cdots$, $0 \le x \le 1$. Prove that $\{f_n\}$ does not converge uniformly on $[0,1]$. *(10%)*

5. (a) Find the extreme values of the real-valued function $f(x_1,\cdots,x_n) = \sum_{i=1}^{n}x_i^2$ under the restriction $\sum_{i=1}^{n}a_ix_i = 1$, where a_i, $i = 1,\cdots,n$ are fixed real numbers, by using the method of Lagrange multipliers. *(10%)*

   (b) Are these extreme values in (a) to be points of local maximum or minimum of f? Justify your answer. *(2%)*

6. A $n\times n$ matrix $\mathbf{X}$ is called idempotent if $\mathbf{X}^2 = \mathbf{X}$. Let $\mathbf{A}$ and $\mathbf{B}$ be $n\times n$ idempotent matrices.

   (a) Suppose $\mathbf{A}-\mathbf{B}$ is idempotent. Prove that $\mathbf{AB} = \mathbf{BA} = \mathbf{B}$. *(8%)*

   (b) Denote $\mathbf{A} = (a_{ij})$ where $a_{ij} = a_{ji}$, $1 \le i, j \le n$. Prove that $\sum_{i=1}^{n}a_{ii} = \dim(\mathbf{A})(= \text{rank}(\mathbf{A}))$. *(10%)*

7. Let $\mathbf{A}$ and $\mathbf{A}^*$ be $n\times n$ matrices. $\mathbf{A}^*$ is said to be a generalized inverse (g-inverse) of $\mathbf{A}$ if $\mathbf{AA^*A} = \mathbf{A}$. Note that $\mathbf{A}^*$ always exists but may not be unique. Assume $\mathbf{G}$ is any g-inverse of $\mathbf{A'A}$.

   (a) Prove that $\mathbf{G}'$ is also a g-inverse of $\mathbf{A'A}$. *(4%)*

   (b) Find a g-inverse of the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 4 & 7 \\ 6 & 4 & 10 \end{bmatrix}$. *(4%)*

   (c) Let $\mathbf{X}$ be an $n\times m$ matrix, and let $\mathbf{P}$ and $\mathbf{Q}$ be $k\times m$ matrices prove that $\mathbf{PX'X} = \mathbf{QX'X}$ implies $\mathbf{PX'} = \mathbf{QX'}$. *(4%)*

   (d) Use (c), to prove $\mathbf{AGA'A} = \mathbf{A}$. *(5%)*

   (e) Prove that $\mathbf{AGA'}$ is symmetric, whether $\mathbf{G}$ is or not. *(5%)*