國立中央大學九十一學年度碩士班研究生入學 卷

所別: 數學系 不分組 科目: 高等微積分 共 1 頁

$\mathbb{R}$ denotes the set of all real numbers.

1. (10%) Assume that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and that $a_n \neq -1$ for all n. Prove that the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is also absolutely convergent.

2. (15%) Let A be the subset of the Euclidean plane $\mathbb{R}^2$ defined by

$$A = \left\{(x,y) : x \in (0,1], y = \sin\frac{1}{x}\right\} \bigcup \left\{(0,y) : y \in [0,1]\right\}.$$

Is A a connected set? Give sufficient reasons to support your answer.

3. (20%) Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. Assume that $\lim_{x\to\infty} f(x) = L$, where L is a finite number. Show that f is uniformly continuous on $[0,\infty)$.

4. (20%) Let

$$f_n(x) = \frac{nx+2}{x^3+n}, \qquad n = 1,2,3,\ldots, \quad x \in [1,3].$$

(a) Find the function $f(x)$ such that $f_n(x) \to f(x)$ for $x \in [1,3]$.

(b) Prove that $f_n(x) \to f(x)$ uniformly on $[1,3]$.

(c) Find $\lim_{n\to\infty} \int_1^3 \frac{nx+2}{x^3+n}\,dx$.

5. Consider the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Let V be an open set in $\mathbb{R}^n$ and $a \in V$, $f : V \to \mathbb{R}^m$.

(a) (5%) Give the definition that f is differentiable at a.

Let $g(x,y) = (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}$ for $(x,y) \neq (0,0)$ and $g(0,0) = 0$.

(b) (10%) Find $g_x(0,0)$ and $g_y(0,0)$.

(c) (10%) Prove that g is differentiable at $(0,0)$.

(d) (10%) We say that f is continuously differentiable at a if all first-order partial derivatives of f exist and are continuous at a.

Is the function g defined above continuously differentiable at $(0,0)$? Prove or disprove it.

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