參考用

1. (28 pts)

(1) Show directly by definition that the interval (0,1) is not compact in $\mathbb{R}$.

(2) Is $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty}\left[\left(\frac{x^{2n}}{(2n)!}+\frac{x^{2n+2}}{(2n+2)!}\right)-\left(\frac{x^{2n+1}}{(2n+1)!}+\frac{x^{2n+3}}{(2n+3)!}\right)\right]$ true? (x is a real number) Justify your answer.

(3) Let $f(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$. Prove $f'(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ for $x\in(-\infty,\infty)$?

(4) Let $f(x)=x$ if x is rational and $f(x)=1-x$ if x is irrational. Show that f is continuous only at $x=\frac{1}{2}$

2. (12 pts)

(1) Show that $\frac{1}{x-2}$ is continuous but not uniformly continuous on $[0,2)$.

(2) If $g(x)$ is differentiable on $[0,2]$ and satisfies $g(2)=g'(2)=0$ then prove that

(i) $\lim_{x\to 2}\frac{g(x)}{x-2}$ exists and (ii) $\frac{g(x)}{x-2}$ is uniformly continuous on $[0,2)$.

3. (12 pts)

(1) Let A be a subset of $\mathbb{R}$ which has an upper bound. Prove: $\sup A\in\overline{A}$

(2) Let f be a continuous real-valued function defined on a compact metric space X. Prove: f assumes its maximum value on X.

4. (12 pts)

Discuss the pointwise and uniform convergence for $x\in(0,1)$, as $n\to\infty$, of the following sequences:

(1) $\{n\,e^{-nx^2}\}$　　(2) $\{(\sin x)\,e^{-nx^2}\}$

5. (12 pts)

Evaluate, justifying all setps, the integral $\int_0^{\infty} e^{-x^2}\sin(2xy)\,dx$.

6. (12 pts)

Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(x,y)=(e^x\cos y, e^x\sin y)$. Show f is invertible at every $(x,y)\in\mathbb{R}^2$, but f has no inverse defined on the whole $\mathbb{R}^2$.

7. (12 pts)

(1) Starting from $\ln(1+x)=\int_0^x\frac{1}{1+t}dt$, show:

$$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n+1}\quad \text{for } |x|<1.$$

(2) Is the following equality true? Justify your results.

$$\ln 2=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n+1}$$