國立中央大學九十三學年度碩士班研究生入學試題卷 共 2 頁 第 / 頁

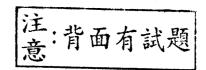
所別:數學系碩士班 不分組科目:線性代數

以下各題,只給答案,沒有說明,不給分

In the following, the symbol $\mathbb R$ denotes the field of real numbers as usual.

- 1. Let n be a positive integer. Let $A=(a_{i,j})$ be an $n\times n$ matrix whose entries $a_{i,j}=i+j-n$ for $i,j=1,2,\ldots,n$. Let $T_A:\mathbb{R}^n\to\mathbb{R}^n$ be the linear transformation $T_A(x)=Ax$ for column vector $x\in\mathbb{R}^n$.
 - (a) (15 \Re) Find the null space $N(T_A)$ and the range $R(T_A)$ of T_A by giving bases for $N(T_A)$ and $R(T_A)$ respectively.
 - (b) (6分) Prove or disprove that det(A) = 0 if and only if $n \ge 3$.
- (c) (9分) Prove or disprove that the charateristic polynomial $f(\lambda)$ of A is of the form $f(\lambda) = (-1)^n (\lambda^n n\lambda^{n-1} + b\lambda^{n-2})$ for some $b \in \mathbb{R}$ such that $b \leq \frac{n^2}{4}$.

 Note. If you can not solve the problem for the general case, you can still get partial credit by verifying the case where n = 4.
 - 2. (12 $\widehat{\beta}$) Let $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}, i = 0, ..., 3\}$ be the set of polynomials of degrees at most 3. Note that P_3 is a vector space over the real numbers. Let $D: P_3 \to P_3$ be the linear operator defined by D(f(x)) = (x+2)f'(x) for all $f(x) \in P_3$. Is D a diagonolizable operator? Explain your answer. If the answer is yes, give an ordered basis β of P_3 such that $[D]_{\beta}$ is a diagonal matrix.
- 3. Let V, W be vector spaces over a common field F and let $\mathcal{L}(V, W)$ denote the set of all linear transformations from V to W. Assume that V is finite dimensional. Fix an operator U on W (that is, $U \in \mathcal{L}(W, W)$). Let $T_U : \mathcal{L}(V, W) \to \mathcal{L}(V, W)$ be the linear transformation $T_U(S) = US$ (composition of U and S) for $S \in \mathcal{L}(V, W)$. Let $f(\lambda)$ be the characteristic polynomial of U. Prove the following statement or disprove it by giving counterexamples.
 - (a) (6 \Re) $f(T_U)=0$, the zero transformation. on $\mathcal{L}(V,W)$.
 - (b) (12 分) Assume that W is of finite dimensional with dim W=m. Let $g(\lambda)$ be the characteristic polynomial of T_U , then $g(\lambda)=f(\lambda)^m$.
- 4. Let n be a positive integer and let $V = M_{n \times n}(\mathbb{R})$. Define $\langle X, Y \rangle = \text{Tr}(Y^t X)$ for $X, Y \in V$, where Y^t denotes the transpose of Y and $\text{Tr}(A) = \sum_{i=1}^n A_{i,i}$ denotes the trace of a matrix A. Fix an $n \times n$ matrix A. and define the linear operator $T_A: V \to V$ by $T_A(X) = AX$ for any $X \in V$.



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- (a) (10 分) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.
- (b) (7分) What is the adjoint operator T_A^* of T_A ? Explain your answer.
- (c) (8分) Let $W = \{X \in V; \text{Tr}(X) = 0\}$. Compute the orthogonal complement W^{\perp} of W by giving an orthonomal basis for W^{\perp} . What is dim W^{\perp} ? Explain your answer.

5. (15
$$\Re$$
) Let $L_i(x_1, x_2, ..., x_n) = \sum_{j=1}^n a_{i,j} x_j, \ a_{i,j} \in \mathbb{R} \text{ for } i = 1, ..., m.$ Let
$$W = \{(b_1, ..., b_n) \in \mathbb{R}^n; L_i(b_1, ..., b_n) = 0\}$$

be the subspace of \mathbb{R}^n determined by the common zeros of the linear functionals L_1, L_2, \ldots, L_m . Let $f(x_1, x_2, \ldots, x_n)$ be a linear functional such that $f(b_1, \ldots, b_n) = 0$ for all $(b_1, \ldots, b_n) \in W$. Prove or disprove that there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$ such that $f = \sum_{i=1}^m \lambda_i L_i$

