國立中央大學八十六學年度碩士班研究生入學試題卷

所別: 數學研究所 不分組 科目: 微分方程 共 1 頁 第 1 頁

In the following $y = y(t)$, $x = x(t)$, $y' = \dfrac{dy}{dt}$, $x' = \dfrac{dx}{dt}$, $\cdots$, $y^{(k)} = \dfrac{d^k y}{dt^k}$

1. Find all solutions of the following: (38%)

   (1) $y''' - 1 = 0$; $y(0) = 0$, $y'(0) = y''(0) = 2$.

   (2) $y''' - 3y^2 + 4y = e^{2t} + e^{-t}$ (3) $(e^y + xe^y)dx + xe^y dy = 0$

   (4) $$\begin{aligned} x' &= x - y \\ y' &= x + y + \cos t \end{aligned}$$

2. Let $L(y) = y' + a(t)y + b(t)$, where a, b are continuous functions on an interval I. Then find the general solution of $L(y) = 0$. Justify your result. (10%)

3. Let $L(y) = t^2 y'' + aty'' + bt$, where a, b are constants. Consider the Euler equation: (15%)

$$L(y) = 0 \qquad \text{for } t \geq 0$$

   Let r_1, r_2 are roots of $q(r) = r(r-1) + ar + b$..

   (1) Show the two linearly independent solutions of this equation are given by $\phi_1(t) = t^{r_1}$, $\phi_2(t) = t^{r_2}$ if $r_1 \neq r_2$ and given by $\phi_1(t) = t^{r_1}$, $\phi_2(t) = t^{r_1} \ln t$ if $r_1 = r_2$.

   (2) Give solutions for the equation: $L(y) = g(t)$ where $g(t)$ is a continuous function on an interval.

4. Let let $W(\phi_1, \cdots, \phi_n)$ be the Wronskian of n solutions $\phi_1, \cdots, \phi_n$ of (15%)

$$L(y) = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0$$

   on an interval I, where a_i $(i = 1, \cdots, n)$ are continuous functions on I. Prove:

   (1) If $W(\phi_1, \cdots, \phi_n)(t) \neq 0$ on I then $\phi_1, \cdots, \phi_n$ are linearly independent on I.

   (2) If $t_0 \in I$ then $W(\phi_1, \cdots, \phi_n)(t) = \exp\left[-\int_{t_0}^{t} a_1(s)ds\right] W(\phi_1, \cdots, \phi_n)(t_0)$ for $t \in I$.

5. Consider a differential equation: $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$ where a_i $(i = 1, \cdots, n)$ are constants. If all roots of characteristic polynomial of $L(y) = 0$ has nonpositive real parts then show that all soltions of $L(y) = 0$ are bounded. (10%)

6. Consider the equation (12%)

$$ty'' + (1-t)y' + y = 0$$

   (1) Find the roots of the indicial polynomial.

   (2) What is the form of the solution of $L(y) = 0$. Find the first four terms of the series of the solution.