國立中央大學八十六學年度碩士班研究生入學試題卷

所別: 數學研究所 不分組 科目: 抽象代數 共 1 頁 第 1 頁

1. (15%) Show that a finite cyclic group of order n has exactly one subgroup of each order d dividing n, and that these are all the subgroups it has.

2. (15%) The **Euler phi-function** is defined for positive integers n by $\phi(n) = s$, where s is the number positive integers less than or equal to n that are relatively prime to n. Show that

$$n = \sum_{d|n} \phi(d),$$

the sum being taken over all positive integers d dividing n.

3. (20%) Prove **Cayley's Theorem**: Every group is isomorphic to a group of permutations.

4. (10%) Show that if p is a prime, then $\mathbf{Z}_p$ has no divisors of 0.

5. (20%) Let $\sigma_m : \mathbf{Z} \to \mathbf{Z}_m$ be the natural homomorphism given by $\sigma_m(a) =$ (the remainder of a when divided by m) for $a \in \mathbf{Z}$.

(a). Show that $\overline{\sigma_m} : \mathbf{Z}[x] \to \mathbf{Z}_m[x]$ given by

$$\overline{\sigma_m}(a_0 + a_1x + \cdots + a_nx^n) = \sigma_m(a_0) + \sigma_m(a_1)x + \cdots + \sigma_m(a_n)x^n$$

is a homomorphism of $\mathbf{Z}[x]$ onto $\mathbf{Z}_m[x]$.

(b). Show that if $f(x) \in \mathbf{Z}[x]$ and $\overline{\sigma_m}(f(x))$ both have degree n and $\overline{\sigma_m}(f(x))$ does not factor in $\mathbf{Z}_m[x]$ into polynomials of degree less than n, then $f(x)$ is irreducible in $\mathbf{Q}[x]$

6. (10%) Show that every maximal ideal of a commutative ring R with unity is a prime ideal.

7. (10%) Prove that if F is a field, every proper nontrivial prime ideal of $F[x]$ is maximal.

END