國立中央大學八十五學年度碩士班研究生入學試題卷

所別：數學研究所 不分組 科目：抽象代數 共 1 頁 第 1 頁

Notations : The positive integers, integers, rational numbers will be denoted by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ respectively.

(1) (a) (10 %) Let G be a group and let $\phi : G \to \mathbb{Z}$ be a group homomorphism from G to the additive group of integers $\mathbb{Z}$. Show that the kernel of ϕ contains the commutator subgroup of G.

(b) (10 %) Let R be a commutative ring with unit element and let $\psi : R \to \mathbb{Z}$ be a ring homomorphism from R to the ring of integers $\mathbb{Z}$. Show that the kernel of ψ is a prime ideal of R. (An ideal I is prime if for any two elements a, b of R, $ab \in I$ implies $a \in I$ or $b \in I$.)

(2) (10 %) Does there exit a field consisting of 1996 elements?
If the answer is Yes, give an example and verify your answer.
If the answer is No, explain why.

(3) (20 %) Let G be a finite group and let H be a subgroup of G. The conjugates of H in G are subgroups of the form $gHg^{-1}, g \in G$. Show that the number of distinct conjugates of H in G is the index of $N(H)$ in G, where

$$N(H) = \{g \in G \,|\, gHg^{-1} = H\}.$$

(4) (15 %) Let R be a commutative ring and let I be an ideal of R. The radical of I is the set

$$\mathfrak{R}(I) = \{r \in R \,|\, r^n \in I \quad \text{for some } n \in \mathbb{N}\}.$$

Show that $\mathfrak{R}(I)$ is an ideal of R.

(5) (15 %) Let S_4 denote the permutation group on 4 elements. How many 2-Sylow subgroups does S_4 have? Determine all the 2-Sylow subgroups of S_4.

(6) (20 %) Let E be a splitting field of the polynomial $(x^2-3)(x^2-5)$. What is the degree $[E : \mathbb{Q}]$ of E over $\mathbb{Q}$? Determine the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ of the polynomial $(x^2-3)(x^2-5)$.