參考用

一、Prove that the distinct equivalence classes of an equivalence relation on a set A provide us with a decomposition of A as a union of mutually disjoint subsets. (10%)

二、Show that G is an abelian (commutative) group if and only if $(ab)^2 = a^2b^2$, $\forall a, b \in G$. ((⇒) 3%, (⇐) 7%)

三、Let G be the group of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$ and a, b, c, d are integers modulo 3, relative to matrix multiplication. Show that the order of G is $o(G) = 48$. (10%)

四、If H and K are subgroups of G with $o(H) > \sqrt{o(G)} < o(K)$, prove that $o(H \cap K) \neq 1$. (10%)

五、Prove that a subgroup N of G is normal if and only if the product of two right cosets of N in G is again a right coset of N in G. ((⇒) 3%, (⇐) 7%)

六、Show that any group of order 72 must have a nontrivial normal subgroup (hence cannot be simple). (10%)

七、Let R be a ring, possibly noncommutative, in which $xy = 0$ implies $x = 0$ or $y = 0$. If $a, b \in R$ with $a^n = b^n$ and $a^m = b^m$ for two relatively prime positive integers m and n, prove that $a = b$. (10%)

八、Let p be a prime number. Prove that the minimal polynomial of $\sqrt[n]{p}\, e^{2\pi\sqrt{-1}/n}$ in $\mathbb{Q}[x]$ is $x^n - p$ where $\mathbb{Q}$ is the field of rationals and n is any positive integer. (10%)

九、Let F be a subfield of a field K and $a \in K$. Prove that a is algebraic over F if and only if there exists a basis of $F(a)$ over F of the form $\{1, a, a^2, \cdots, a^n\}$ where $F(a)$ is the smallest subfield of K containing both F and a. ((⇐) 3%, (⇒) 7%)

十、Find the Galois group of $x^5 - 6x + 3$ over $\mathbb{Q}$. (10%)

參考用