所別：通訊工程學系碩士班 通訊網路組 科目：工程數學

注意: 本試題共分三部份 (每部份 50分) 請考生任選兩部分作答

PART I 線性代數 (50分)

1.(13%) Let V denote the inner product space of real-valued continuous functions on $[0, T]$. For $r(t), s_1(t), s_2(t),$
$\cdots, s_M(t) \in V$, we want to find $s_i(t)$ $(1 \leq i \leq M)$ which has the minimum value of $\|r(t) - s_i(t)\|$, the distance between $r(t)$ and $s_i(t)$.

(a)(6%) Suppose that $\|s_1(t)\| = \|s_2(t)\| = \cdots = \|s_M(t)\|$. Prove that $s_i(t)$ which minimizes $\|r(t) - s_i(t)\|$ also maximizes the inner product $\langle r(t), s_i(t) \rangle$.

(b)(7%) Let $\beta = \{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$ represent an ordered orthonormal basis for V. Let the coordinate vector of $r(t)$ and $s_i(t)$ relative to β be $[r(t)]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $[s_i(t)]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, respectively.

Prove that $\|r(t) - s_i(t)\|^2 = \sum_{j=1}^{n} |a_j - b_j|^2$.

2.(14%) Let $P(R)$ denote the vector space which consists of all polynomials with coefficients from R, the field of real numbers. Let W be the set of all polynomials $f(x)$ in $P(R)$ such that in the representation $f(x) = a_{4n-1}x^{4n-1} + a_{4n-2}x^{4n-2} + a_{4n-3}x^{4n-3} + \cdots + a_1 x + a_0$ where n is a positive integer, we have $a_i = 0$ whenever i is even (including 0) and $\sum_{j=1}^{n} a_{4j-1} = \sum_{j=1}^{n} a_{4j-3} = 0$.

(a)(7%) Is W a subspace of $P(R)$? Justify your answer.

(b)(7%) If the answer for (a) is "Yes", find a basis for W; otherwise find the smallest subspace that contains W.

3.(10%) Recall the definition of $P(R)$ in the previous question. Define $T : P(R) \to P(R)$ by $T(f(x)) = \int_0^x f(t)dt$ and define $U : P(R) \to P(R)$ by $U(f(x)) = f'(x)$. Determine whether UT (first T then U) and TU are one-to-one or onto. Justify your answer.

4. (a)(6%) Evaluate the determinant of $\begin{bmatrix} 6 & 1 & -1 & 5 \\ 2 & -1 & 3 & -2 \\ 1 & 0 & -1 & 0 \\ -4 & 3 & 2 & 1 \end{bmatrix}$.

(b)(7%) Find all solutions to the system $2x_1 - x_2 + x_3 = 2$ and $x_1 + 3x_2 - x_3 = 1$.

注意：背面有試題

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PART II 機率 (50分)

1. (12%) Let $\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}\,dx$. Show that

$$\Gamma(n+1/2)=\frac{(2n)!}{n!2^{2n}}\sqrt{\pi}, \qquad n=0,1,2,\ldots$$

2. (13%) Let X have the beta distribution, which has a probability density function

$$f_X(x)=Cx^{\alpha-1}(1-x)^{\beta-1}, \qquad 0<x<1,$$

where C is the normalization constant. Determine the probability density function of the random variable $X^{-1}-1$.

3. (10%) A quiz was administered to four students. Somehow the quizzes got shuffled, and the one at the top of the stack was returned to the first student, the one below it was returned to the second student, and so on. Find the probability that at least one student got his own quiz back.

4. (15%) An urn contains N balls, identical in every respect except that they carry numbers $(1,2,\ldots,N)$ and M of them are colored red, the remaining $(N-M)$ white, $0\le M\le N$. We draw a ball from the urn blindfolded, observe and record its color, lay it aside, and repeat the process until n balls has been drawn, $0\le n\le N$.

(a) (7%) Find the probability of red on the first r consecutive draws in a specified order.

(b) (8%) Find the probability of red on the third draw.

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PART III. 離散數學 (50分)

1. (10%) A Graph G = (V,E) is called bipartite if $V = V1 \cup V2$ with $V1 \cap V2 = \varnothing$, and every edge of G is of the form {a,b} with $a \in V1$ and $b \in V2$. If each vertex in V1 is joined with every vertex in V2, we have a complete bipartite graph. In this case, if $|V1|=m$, $|V2|=n$, the graph is denoted as K(m,n). How many paths of longest length are there in a complete bipartite graph K(m,n) ? (Remember that a path such as $v1 \to v2 \to v3$ is considered as the same as the path $v3 \to v2 \to v1$)

2. (10%) Let G =(V,E) be an undirected graph with adjacent matrix A(G) as shown below.

	v1	v2	v3	v4	v5	v6	v7	v8
v1	0	1	0	0	0	0	1	0
v2	1	1	0	1	1	0	1	0
v3	0	0	0	1	0	1	0	1
v4	0	1	1	0	0	0	0	0
v5	0	1	0	0	0	0	1	0
v6	0	0	1	0	0	1	0	0
v7	1	1	0	0	1	0	0	0
v8	0	0	1	0	0	0	0	0

Use a breath-first search based on A(G) to determine whether G is connected.

3. (10%) What is the maximum number of internal vertices that a complete quaternary tree of height 8 can have?

4. (10%) Solve the recurrent relation $a_n + a_{n-1} - 6a_{n-2} = 0$, where $n \geq 2$ and $a_0 = 1$, $a_1 = 2$.

5. (a) Write a computer program (or develop an algorithm) to locate the first occurrence of the maximum value in an array $A[1], A[2], A[3], \ldots, A[n]$ of integers. (Here $n \in Z^+$ and the entries in the array need not be distinct.) (5%)
   (b) Determine the worse-case complexity function for the implementation developed in part (a). (5%)