Discrete Math.

APR. 14, 1996

Instructions. 以下六題請任選五題作答，每題20分。也可以六題都做，閱卷老師會酌情予以加分，但滿分仍爲100分。以下各題若非證明題，仍須將運算及分析過程寫出。

1. Prove or disprove the following statements:

   (a) $n^2 - n + 41$ is prime for $n = 0, 1, 2, ...$

   (b) $1(1!) + ... + n(n!)$ can be expressed as difference of two factorials for $n = 1, 2, 3, ...$ (we assume $0! = 1$).

2. Let $S = \{(i,j) |\ i, j$ are positive integers and $1 \leq i, j \leq 5\}$. A binary relation R on S is defined as follows: $(i_1, j_1)\, R\, (i_2, j_2)$ if $i_1 + j_2 = i_2 + j_1$

   (a) Show that R is an equivalence relation.

   (b) Find the corresponding partition of S.

3. Both Dijkstra's algorithm and dynamic programming can be used to solve the shortest path problem in a weighted directed graph.

   (a) Give an example that Dijkstra's algorithm doesn't work.

   (b) Give an example that dynamic programming doesn't work.

4. Let a_r be the number of subsets of the set $\{1, 2, ...r\}$ that do not contain two consecutive numbers (two integers i and j are said to be consecutive if $j = i + 1$ or $j = i - 1$.) Determine a_r. (Hint: Consider the two disjoint collections of subsets that contain or do not contain r respectively, and then derive a recurrence relation of a_r.)

5. Consider an undirected graph with n vertices and $n - 1$ edges. Assume that the input format of the graph is: the number n and then followed by $n - 1$ pairs of integers (i, j), where each (i, j) means that there is an edge joints vertices i and j. Design a linear time algorithm (i.e. the execution time of your algorithm must be in $O(n)$) to determine whether the given graph is a tree.

6. Suppose that n balls are tossed into n bins, where each toss is independent and the ball is equally likely to end up in any bin. What is the expected number of empty bins? What is the expected number of bins with exactly one ball? (Hint: define some appropriate random variables $X_1, X_2..., X_n$, and then use the formula $E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$)