類組：電機類　科目：工程數學D(3006)　※請在答案卷內作答

一、(10%) Find a differentiable function $f(t)$ with $f(0)=0$ such that the differential equation

$$y^2 \sin t + y f(t)\frac{dy}{dt} = 0$$

is exact. Solve the equation.

二、(10%) Find the general solution of the differential equation

$$y'' - 6y' + 9y = (1 + t + t^2 + \cdots + t^{15})e^{3t}.$$

三、(10%) Consider the initial value problem

$$x'' + p_0 x' + q_0 x = f(t),\ t \ge 0,\ x(0)=0,\ x'(0)=0.$$

(一) (5%) Determine $p_0$ and $q_0$ so that the solution $x(t)$ can be expressed as

$$x(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau) f(\tau)\, d\tau,\ t \ge 0.$$

(二) (5%) Compute $x(t)$ if $f(t) = 2\delta(t-\pi)$.

四、(10%) Solve the initial-value problem

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t,\ \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

五、(10%) Find a formal Fourier series solution of the endpoint value problem

$$x'' + 4x = 4t,\ x'(0) = x'(1) = 0..$$

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六、(10%) Find the solutions to the linear systems $A\mathbf{x} = \mathbf{b}$ with the following rref (reduced row echelon form).

(Note: write "No solution" if it has no solution.)

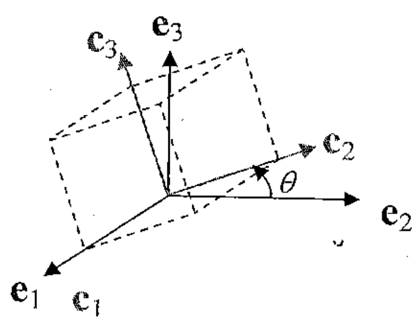
1. $\text{rref}([A \mid \mathbf{b}_1]) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 & 3 \\ 0 & 0 & 1 & 0 & -2 \end{array}\right]$ 2. $\text{rref}([A \mid \mathbf{b}_2]) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$

3. $\text{rref}([A \mid \mathbf{b}_3]) = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$ 4. $\text{rref}([A \mid \mathbf{b}_4]) = \left[\begin{array}{ccccc|c} 0 & \mathbf{1} & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

七、(10%) Find the transformation matrices $A$ of the following linear transformations $\mathbf{y} = T(\mathbf{x})$.

(一)、(5%)

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 10 \\ 1 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$$




(二)、(5%) The rotation about $\mathbf{e}_1$ is shown in the right figure. This rotation is a transformation $T: \Re^3 \rightarrow \Re^3$, such that

$T(\mathbf{e}_1) = \mathbf{c}_1$, $T(\mathbf{e}_2) = \mathbf{c}_2$, $T(\mathbf{e}_3) = \mathbf{c}_3$,

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\Re^3$. According to trigonometry,

$$\mathbf{c}_1 = \mathbf{e}_1,\ \mathbf{c}_2 = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix},\ \mathbf{c}_3 = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Find $A$ such that of $T(\mathbf{x}) = A\mathbf{x}$.

八、(一)、(4%) Suppose $A$ is a square matrix and $A = A^T$. Show that

$$\mathbf{v}^T A\, \mathbf{v} \le \lambda_{\max} \|\mathbf{v}\|^2 \text{ for any vector } \mathbf{v},$$

where $\lambda_{\max}$ is the maximum value of the eigenvalues of $A$.

(**Hint**: An $n \times n$ square matrix $A$ is orthogonally diagonalizable if and only if $A$ is symmetric.)

(二)、(6%) Suppose that the transformation matrix of $T$ is $C$, i.e., $T(\mathbf{x}) = C\,\mathbf{x}$. It is known that vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ are eigenvectors of matrix $C$ associated with the eigenvalues 18, 10, 4, -12, respectively.

A basis (or **eigenbasis**) $B$ for $C$ is $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$.

$$C = \begin{bmatrix} 5 & 2 & 9 & -6 \\ 2 & 5 & -6 & 9 \\ 9 & -6 & 5 & 2 \\ -6 & 9 & 2 & 5 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix},\ \mathbf{b}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Find the transformation matrix $[C]_B$ for $T$ w.r.t. basis $B$, i.e., $[C]_B$=?

(**Notation** definition for $[C]_B$: if $\mathbf{y} = C\mathbf{x}$, then $[\mathbf{y}]_B = [C]_B\,[\mathbf{x}]_B$, where $[\mathbf{x}]_B$ denotes the coordinate vector of $\mathbf{x}$ relative to basis $B$.)

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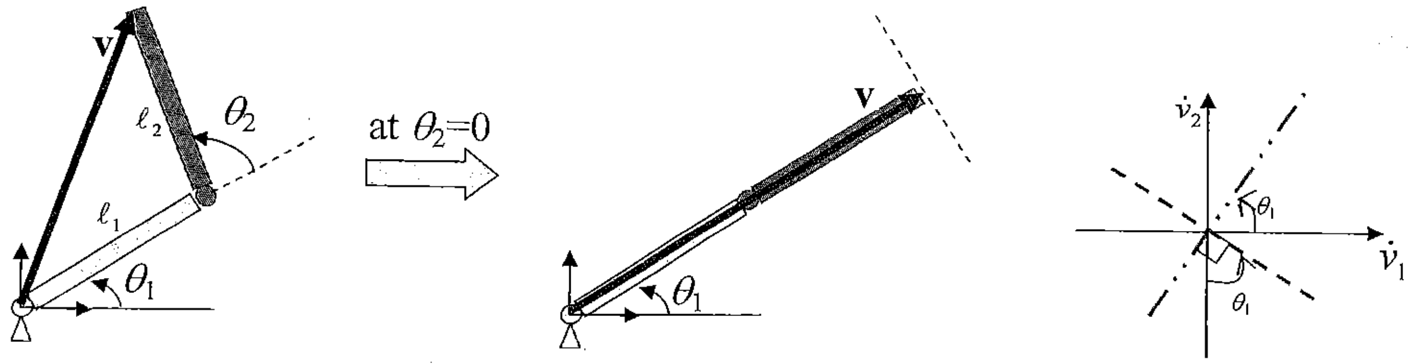
九、(10%) Consider a Scara robot (or called two-link robot) shown in the following figure.




The relationship between velocity vector $\dot{\mathbf{v}}$ and joint velocities $\dot{\theta}_1, \dot{\theta}_2$ is the Jacobian matrix $J(\theta_1,\theta_2)$:

$$\dot{\mathbf{v}} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = J(\theta_1,\theta_2)\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \quad \text{where } J(\theta_1,\theta_2) = \begin{bmatrix} \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1+\theta_2) & \ell_2 \sin(\theta_1+\theta_2) \\ -\ell_1 \cos\theta_1 - \ell_2 \cos(\theta_1+\theta_2) & -\ell_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

When at a singular configuration (i.e., $\theta_2$=0), $J(\theta_1,\theta_2)$ with $\ell_1 = 1, \ell_2 = 1$, is reduced to

$$J(\theta_1,\theta_2=0) = \begin{bmatrix} 2\sin\theta_1 & \sin\theta_1 \\ -2\cos\theta_1 & -\cos\theta_1 \end{bmatrix}, \text{ and}$$

$$\text{rref}\big(J(\theta_1,\theta_2=0)\big) = \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}, \qquad \text{rref}\big(J^T(\theta_1,\theta_2=0)\big) = \begin{bmatrix} 1 & -\dfrac{\cos\theta_1}{\sin\theta_1} \\ 0 & 0 \end{bmatrix}$$

(一)、1. (2%) Find a basis of range($J(\theta_1,\theta_2$=0))={**w**}, i.e., **w** =? (note: range = column space)

2. (1%) Draw the range($J(\theta_1,\theta_2$=0)) on the $\dot{v}_1$-$\dot{v}_2$ plane.

(二)、1. (2%) Find the orthogonal complement of range($J$) for $\theta_2$=0, i.e., ker($J^T$)= span(**h**) and **h** =? (note: ker = kernel = null space)

2. (1%) Draw the vector **h** on the $\dot{v}_1$-$\dot{v}_2$ plane.

(三)、Let **w** and **h** be those obtained above .

1. (2%) Does $J(\theta_1,\theta_2=0)\mathbf{x} = (3\mathbf{h} - \mathbf{w})$ have solution(s) **x**?
2. (2%) Find the orthogonal projection matrix $P$ of $\text{proj}_{\text{span}(\mathbf{w})}$, and verify it by $P\,\mathbf{h} = \mathbf{0}$.

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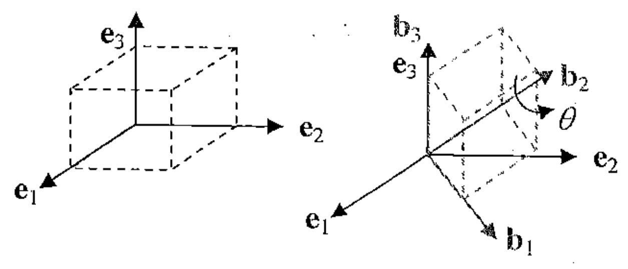
十、(一)、(6%) Find the rotation matrix, with respect to the standard basis, about the axis of $\mathbf{b}_2$ by $\theta$, where

$$\mathbf{b}_1=\begin{bmatrix}0.6\\0.8\\0\end{bmatrix},\quad \mathbf{b}_2=\begin{bmatrix}-0.8\\0.6\\0\end{bmatrix},\quad \mathbf{b}_3=\mathbf{e}_3.$$




It is known that the rotation matrix about $\mathbf{e}_2$ is

$$Rot(\mathbf{e}_2,\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}$$

(**Hint**: Similarity transformation)

(二)、(4%) Write the contra-positive statements of the following statements. ("If not B, then not A" is a contra-positive form of "if A, then B")

1. If the columns of $A\in\Re^{m\times n}$ are linearly independent, then $n\le m$.
2. $n$ nonzero vectors are linearly independent, if they are orthogonal.
3. If 0 is not an eigenvalue of $A$, then $\det A\ne 0$.
4. For any matrices $A\in\Re^{m\times n}$ and $B\in\Re^{n\times k}$, if the columns of $B$ are linearly dependent, then those of $AB$ are linearly dependent.

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