國立中央大學106學年度碩士班考試入學試題

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科目： 應用數學

本科考試禁用計算器　　須有計算過程　　*請在答案卷內作答

參考用

1. (Total 15%) Please solve the following differential equations

(i) (5%) $\frac{dx(t)}{dt}+\alpha x(t)=H(t)$, where $H(t)=\begin{cases}H_0 \text{ for } t\geq 0\\ 0 \text{ for } t<0\end{cases}$, α, H_0 are constants

and the initial condition $x(0)=0$. Find $x(t)$

(ii) (5%) $\begin{cases}\frac{dx(t)}{dt}=-\omega y(t)\\ \frac{dy(t)}{dt}=\omega x(t)\end{cases}$ where ω is a constant and the initial condition

$x(0)=1, y(0)=0$. Find $x(t)$ and $y(t)$

(iii) (5%) $\begin{cases}\frac{\partial U(x,y)}{\partial x}=2xy-1\\ \frac{\partial U(x,y)}{\partial y}=x^2+2y\end{cases}$ and $U(0,0)=0$. Find $U(x,y)$

2. (Total 20 %)

(i) (10%) Find the unit vector perpendicular to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at the point

$$(x,y,z)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$$

(ii) (10%) Derive the equation of the plane tangent to the surface at

$$(x,y,z)=\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$$

3. (Total 10%) Calculate

(i) (5%) $\frac{1}{N}\sum_{k=1}^{N}\left(\cos\frac{2\pi k}{N}\right)^2$

(ii) (5%) $\frac{1}{N}\sum_{k=1}^{N}\left[\cos\left(\frac{2\pi k}{N}\right)\cdot\sin\left(\frac{2\pi k}{N}\right)\right]$

注意：背面有試題

4. (Total 20%)

(i) (10%) If a matrix M has eigenvector A and eigenvalue λ, show that its inverse M^{-1} has eigenvector A and eigenvalue $1/\lambda$

(ii) (10%) A unitary matrix U is defined as $U^{\dagger}=U^{-1}$ where $\left(U^{\dagger}\right)_{ij}=U_{ji}^{*}$, if λ is an eigenvalues of U, show that $|\lambda|=1$

5. (Total 20%) A periodic function of period 2π can be expanded as Fourier series as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

(i) (5%) If $f(x)$ is a periodic function of period 2π and $f(x)=x,\quad -\pi<x\le\pi$, expand $f(x)$ as Fourier series

(ii) (10%) Prove that for any periodic function of period 2π

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)\right]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

(iii) (5%) From (i) and (ii) show that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

6. (Total 15%) The Poisson probability distribution is described as $p(x)=\frac{e^{-a}a^x}{x!}$ where $x=0,1,2,\cdots,\infty$ and a is a constant. The expectation value of $f(x)$ is defined as $\langle f(x)\rangle=\sum_{x=0}^{\infty}f(x)p(x)$, please calculate

(i) (5%)The mean value, $\langle x\rangle$

(ii) (10%)The variance, $\left\langle\left(x-\langle x\rangle\right)^2\right\rangle$