國立中央大學109學年度碩士班考試入學試題

所別： 天文研究所 碩士班 不分組(一般生)　　共2頁 第1頁

科目： 應用數學

本科考試禁用計算器　＊計算題需計算過程，無計算過程者不予計分　＊請在答案卷(卡)內作答

1. (15%) Calculate

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \cos(kx)\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \sin(kx)\,dx$$

2. (Total 20%) Calculate

(i) (10%) $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$

(ii) (10%) $\sum_{n=2}^{\infty} \frac{1}{(n-1)n(n+1)}$

3. (Total 15%) Find the general solutions of the following differential equations

(i) (5%) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

(ii) (5%) $x^2\frac{d^2y}{dx^2} - 6x\frac{dy}{dx} + 6y = 0$

(iii) (5%) $2xy\frac{dy}{dx} + y^2 - 2x = 0$

4. (Total 15%) Suppose f and g are differentiable functions of three dimensional space, show that

(i) (5%) $\nabla(fg) = f\nabla g + g\nabla f$

(ii) (5%) $\nabla\cdot[\nabla(fg)] = f\nabla^2 g + 2(\nabla f)\cdot(\nabla g) + g\nabla^2 f$

(iii) (5%) $\nabla\times(\nabla f) = 0$

參考用

注意：背面有試題

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5. (Total 15%) For a complex number $z = x + iy$ where x and y are real numbers. Show that

(i) (5%) $\sin(z) = \sin x \cosh y + i \cos x \sinh y$

(ii) (5%) $|\cosh(z)|^2 = \sinh^2 x + \cos^2 y = \cosh^2 x - \sin^2 y$

(iii) (5%) $\tanh(z) = \dfrac{\sinh(2x) + i\sin(2y)}{\cosh(2x) + \cos(2y)}$

6. (Total 20%) The three Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that

(i) (5%) $\sigma_a^2 = I$ where I is the 2×2 unit matrix,

(ii) (5%) $\sigma_a \sigma_b = i\sigma_c$ where $(a, \ b, \ c) = (1,2,3)$ or $(2,3,1)$ or $(3,1,2)$

(iii) (5%) $\sigma_a \sigma_b = -\sigma_b \sigma_a$ if $a \neq b$

(iv) (5%) $(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} I + i\vec{\sigma} \cdot (\vec{A} \times \vec{B})$ where $\vec{A}, \vec{B}$ are ordinary vectors and $\vec{\sigma} = \sigma_1 \hat{x} + \sigma_2 \hat{y} + \sigma_3 \hat{z}$.