＊選擇題請在答案卡內作答。非選擇題請在答案卷內作答。

複選題

Problem 1. (25%) **Multiple Choice Questions:** There may be more than one correct choice for each of the multiple choice questions. You need to select all choices that apply for each question. No credit is given unless you select all of the correct choices and no others.

1. (7%) Consider the following system of linear equations:

$$\begin{aligned} x_1 + 3x_2 + 2x_4 &= b_1 \\ x_3 + 4x_4 &= b_2 \\ x_1 + 3x_2 + x_3 + 6x_4 &= b_3 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \Rightarrow \mathbf{Mx} = \mathbf{b}.$$

Which of the following statements is/are true?

(A) The system is always solvable.
(B) The dimension of the column space of $\mathbf{M}$ is 2.
(C) The dimension of the row space of $\mathbf{M}$ is 2.
(D) The dimension of the nullspace of $\mathbf{M}$ is 2.
(E) The nullspace of $\mathbf{M}$ is orthogonal to the column space of $\mathbf{M}$.

2. (7%) Consider the following matrices:

$$\mathbf{M} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \text{ and } \lim_{k\to\infty} \mathbf{M}^k = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Which of the following statements is/are true?

(A) $a = 3/5$.
(B) $b = 2/5$.
(C) $c = 3/5$.
(D) $d = 2/5$.
(E) None of the above.

3. (6%) Which of the following subsets of $\mathbb{R}^3$ are subspaces (or is a subspace)?

(A) All vectors $[x\ y\ z]^T$ with $x = y$.
(B) All vectors $[x\ y\ z]^T$ with $xyz = 0$.
(C) All vectors $[x\ y\ z]^T$ with $x \le y \le z$.
(D) All vectors $[x\ y\ z]^T$ with $x - y + z = 1$.
(E) All vectors $[x\ y\ z]^T$ with $x - y + z = 0$.

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4. (5%) Consider the following vectors:

$$\mathbf{b} = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix}-1\\2\\2\end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix}2\\2\\-1\end{bmatrix},\ \text{and } \mathbf{a}_3 = \begin{bmatrix}2\\-1\\2\end{bmatrix}.$$

Project $\mathbf{b}$ onto lines through $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ with orthogonal projections $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, respectively. Let $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = [x\ y\ z]^T$ with length (norm) $\|\mathbf{p}\|$. Which of the following statements is true?

(A) $\|\mathbf{p}\| = 1/9$.
(B) $\|\mathbf{p}\| = 1/3$.
(C) $\|\mathbf{p}\| = 2/3$.
(D) $\|\mathbf{p}\| = 1$.
(E) None of the above.

計算題應詳列計算過程，無計算過程者不予計分

Problem 2. (10 %) Let $\mathbb{R}^n$ denote the set of all real $n \times 1$ vectors, and $\text{span}[\mathbf{v}_1, \ldots, \mathbf{v}_k] = \{\sum_{i=1}^k \alpha_i \mathbf{v}_i \mid \mathbf{v}_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}\ \forall i\}$. Suppose

$$W_1 = \text{span}\left[\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right], W_2 = \text{span}\left[\begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right],$$

Find a basis for $W_1 \cap W_2$ (i.e., the intersection of W_1 and W_2). (You need to provide logical reasoning and/or derivations, otherwise no credits)

Problem 3. (15 %) Let

$$\mathbf{A} = \begin{bmatrix}4 & 2 & 2\\2 & 4 & 2\\2 & 2 & 4\end{bmatrix}$$

Find (a) (10%) eigenvalues and orthonormal eigenvectors of $\mathbf{A}$, and (b) (5%) a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. (You need to provide logical reasoning and/or derivations, otherwise no credits.)

Problem 4. (10%) Suppose that A_n is an event with probability 1, i.e., $P(A_n) = 1$, for all $n \geq 1$. Please find $P(\cap_{n=1}^{\infty} A_n)$. (Hint: Boole's inequality $P(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} P(E_n)$ might be useful.)

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Problem 5. (15%) Consider a "one-person jury" consisting of only one juror, say juror 0, and a "three-person jury" consisting of three jurors, say juror 1, juror 2, and juror 3. Assume that juror i makes the correct decision with probability p_i, where $0 < p_i < 1$, for $i = 0, 1, 2, 3$. Also assume that the jurors in the three-person jury decide independently and the decision of the *majority* is final. When is the one-person jury preferable to the three-person jury, namely, when is the probability of making a correct decision by the one-person jury is higher than that by the three-person jury (please express your answer in terms of p_0, p_1, p_2 and p_3)?

Problem 6. (15%) Consider an urn containing a large number of coins, and suppose that each of the coins has some probability p of turning up head when flipped. However, this value of p varies from coin to coin. Suppose that the p-value of a randomly chosen coin can be regarded as being the value of a random variable that is distributed over $[0, 1]$ with a probability density function

$$f(x) = \frac{3}{2}x(2 - x).$$

(a) (7%) What is the probability that the randomly chosen coin turns up head when flipped ?

(b) (8%) What is the conditional expectation of the p-value given that the randomly chosen coin turns up head when flipped ?

(You need to provide logical reasoning and/or derivations, otherwise no credits.)

Problem 7. (10%) Consider a sequence of independent random pairs (X_i, Y_i), $i = 1, 2, \ldots$, each pair having the same joint probability mass function,

$$p(1,1) = 0.2,\ p(1,-1) = 0.3,\ p(-1,1) = 0.3,\ p(-1,-1) = 0.2.$$

Let $S_n = \sum_{i=1}^{n} X_i Y_i$. What is the limit of $\frac{S_n}{n}$ as $n \to \infty$? (You need to provide logical reasoning and/or derivations, otherwise no credits.)