1. (15 pts) Let X_1, X_2 be independent and identically distributed RVs from $N(0,1)$. Set $Y_1 = X_1/X_2$ and find the pdf of Y_1.

2. Let $X_i, i = 1 \ldots, n$ be independent RVs such that X_i has continuous and strictly increasing CDF F_i. Let $Y_i = F_i(X_i)$

   (a) (10 pts) Show Y_i be independent $U(0,1)$.

   (b) (15 pts) Show that

   $$Z = -2 \sum_{i=1}^{n} \ln(1 - Y_i)$$

   is distributed as χ^2_{2n}. Hint: $X \sim \chi^2_2 \iff X \sim Exp(\frac{1}{2})$

3. (10 pts) The moment generating function of random variable X is defined as follows:

   $$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

   , where $f(x)$ is the pdf of X. Show that

   $$E[X^2] = \frac{d^2 M_X(t)}{dt^2} \Big|_{t=0}$$

4. Suppose a civil engineer wishes to compare the strengths of two different types of beams, one (A) made of steel and the other (B) made of a different and more expensive alloy. A certain deflection (in units of 0.001 inch) was measured for each beam when submitted to a given force; thus a small deflection would indicate a beam of great strength. The order observations for the two samples of size $n_1 = 8$, and $n_2 = 6$ are the following:

Types	Deflection (in 0.001 inch)							
A	79	81	83	84	86	86	86	87
B	74	75	76	77	78	82		

   (a) (10 pts) Please conduct a 2-samples t-test to identify whether there is a difference between the two means of deflections of different types of beams? Use $\alpha = 0.05$.

   Suppose there are two more different types of beams, (C) and (D), made of different alloys. The resulted sample means and sample variances of deflections from the two corresponding samples of Type C with size $n_3 = 6$, and of Type D with size $n_4 = 6$, are as following:

Types	Sample size	Sample mean	Sample Variance
C	6	79	7.6
D	6	73	8.0

   (b) (10 pts) Please conduct an ANOVA to identify whether there is a difference between the four means of deflections of different types of beams? Use $\alpha = 0.05$, and summarize the test result in an ANOVA table.

(c) (10 pts) Following the test result in (b), we need to make comparisons between the four means, noted as pairwise-t tests. Please conduct a pairwise-t test between the two means of deflections of type A and type B with $\alpha = 0.05$. Show the difference between (a) and (c).

5. Following 10 records are used to train a <u>Naïve Bayesian Classification</u> model to predict tennis-playing or not, based on weather information.

Day	Outlook	Temperature	Humidity	Wind	Play Tennis
1	Sunny	Hot	High	Weak	*No*
2	Sunny	Hot	High	Strong	*No*
3	Overcast	Hot	High	Weak	*Yes*
4	Rainy	Mild	High	Weak	*Yes*
5	Rainy	Cool	Normal	Weak	*Yes*
6	Rainy	Cool	Normal	Strong	*No*
7	Overcast	Cool	Normal	Strong	*Yes*
8	Sunny	Mild	High	Weak	*No*
9	Sunny	Cool	Normal	Weak	*Yes*
10	Rainy	Mild	Normal	Weak	*Yes*

(a) (4 pts) Let's define the event A as "play_tennis = *yes*", while A^C as "play_tennis = *no*". Please determine the (**prior**) probabilities of $\Pr(A)$ and $\Pr(A^C)$ by <u>counting the frequencies</u> based on the 10 records.

(b) (4 pts) Let's define the event B_1 as "outlook= *sunny*", B_2 as "outlook= *overcast*", and B_3 as "outlook= *rainy*". Please determine the <u>conditional probabilities</u> of $\Pr(B_i|A)$ and $\Pr(B_i|A^C)$, $i = 2, 3$, by counting the frequencies based on the 10 records. (for example, $\Pr(B_1|A)=1/6$ and $\Pr(B_1|A^C)=3/4$)

Similarly, we can define events

C_1 as "temperature = *hot*", C_2 as "temperature = *mild*", and C_3 as "temperature = *cool*";

D_1 as "humidity = *high*", and D_2 as "humidity = *normal*";

E_1 as "wind = *weak*", and E_2 as "wind = *strong*".

Assume events $\{B_i\}$, $\{C_j\}$, $\{D_k\}$, and $\{E_l\}$ are **mutually independent**. Hence the $\Pr(B_i \cap C_j \cap D_k \cap E_l)$ can be determined as $\Pr(B_i)\cdot\Pr(C_j)\cdot\Pr(D_k)\cdot\Pr(E_l)$. For example,

$\Pr$("outlook = *sunny*", "temperature = *cool*", "humidity = *normal*", "wind = *weak*")

$= \Pr$("outlook = *sunny*")$\cdot\Pr$("temperature = *cool*")$\cdot\Pr$("humidity = *normal*")$\cdot\Pr$("wind = *weak*")

$= \dfrac{4}{10} \cdot \dfrac{4}{10} \cdot \dfrac{5}{10} \cdot \dfrac{7}{10} = 0.056$

(c) (4 pts) Let's use event X as "outlook = *sunny*, temperature = *cool*, humidity = *normal*, wind = *weak*". Please determine the <u>conditional probabilities</u> of $\Pr(X|A)$ and $\Pr(X|A^C)$ by counting the frequencies based on the 10 records with assuming independence.

(d) (6 pts) Try to use <u>Bayes' Theorem</u> to calculate the (**posterior**) probabilities, $\Pr(A|X)$ and $\Pr(A^C|X)$.

(e) (2 pts) With the posterior probabilities in (d), would the unknown sample X be classified/predicted as "play_tennis = *yes*" or "play_tennis = *no*"?
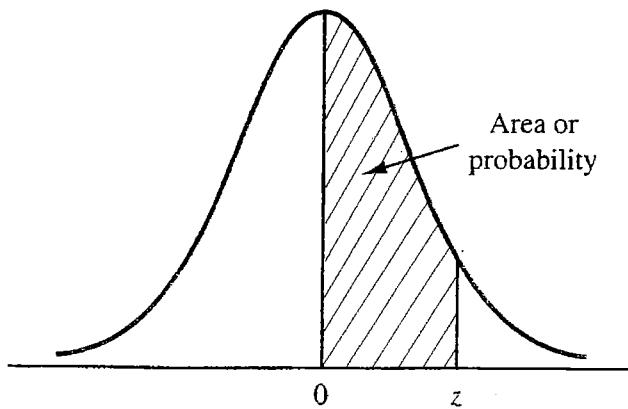
STANDARD NORMAL DISTRIBUTION

Entries in the table give the area under the curve between the mean and z standard deviations above the mean. For example, for $z = 1.25$ the area under the curve between the mean and z is .3944.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4986	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
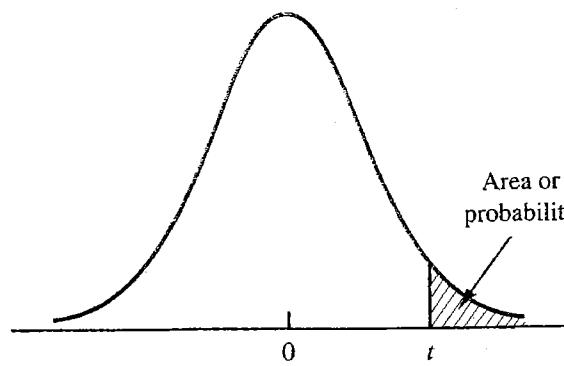
t DISTRIBUTION

Entries in the table give t values for an area or probability in the upper tail of the t distribution. For example, with 10 degrees of freedom and a .05 area in the upper tail, $t_{.05} = 1.812$.

Degrees of Freedom	Area in Upper Tail				
	.10	.05	.025	.01	.005
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.326	2.576
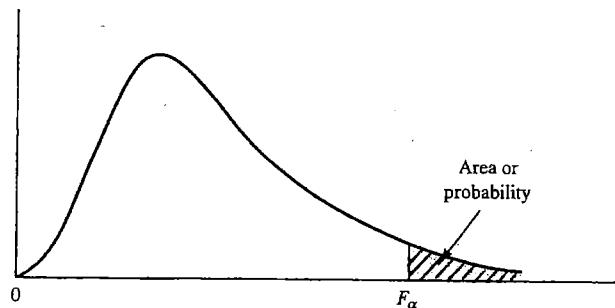
F TABLE - I

Area or probability

Entries in the table give F_α values, where α is the area or probability in the upper tail of the F distribution. For example, with 12 numerator degress of freedom, 15 denominator degrees of freedom, and a .05 area in the upper tail, $F_{.05} = 2.48$.

Table of $F_{.05}$ Values

Denominator Degrees of Freedom	Numerator Degrees of Freedom																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00