多重選擇題，共 20 題，每題 5 分 答錯一個選項倒扣 1 分，倒扣至本大題(即多選題)0 分為止。

1. If a linear transformation from a vector space V to another vector space W is one-to-one, which of the following statements is/are true?
   (A) It is onto;
   (B) dim(V) = dim(W);
   (C) The null space of this transformation contains only the zero vector;
   (D) It is invertible;
   (E) All of the above.

2. Consider a subset $S = \{ (1, 2, 1), (2, -1, 1)\}$ of R^3, which of the following statements is/are true?
   (A) The set S spans a subspace in R^3;
   (B) Such a subspace is the z = 1 plane in R^3;
   (C) The set S is a linearly independent set;
   (D) The two vectors in S are orthogonal in the subspace;
   (E) If we add another vector (0, 0, 1) into S, this set can span R^3.

3. Which of the following processes is/are linear?
   (A) Fourier transform of real-valued functions on R;
   (B) Laplace transform of real-valued functions on R;
   (C) Determinant calculation of real $n \times n$ matrices;
   (D) Inner product of an arbitrary vector $\boldsymbol{x}$ and a fixed vector $\boldsymbol{z}$ in R^n;
   (E) Norm calculation of vectors in R^n.

4. What is the rank of the matrix:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}?$$

   (A) 0;
   (B) 1;
   (C) 2;
   (D) 3;
   (E) 4.

5. Find the determinant of the symmetric Pascal matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

   (A) 0;
   (B) 1;
   (C) 2;
   (D) 3;
   (E) 4.

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6. Find the parabola $y = a + b\,x + c\,x^2$ that comes closest (least squares error) to the data points: $(x, y) = (-2, 0), (-1, 0), (0, 1), (1, 2)$, and $(2, 0)$.
   (A) $a = 40/35, b = 0, c = 1/7$;
   (B) $a = 41/35, b = 1/5, c = -2/7$;
   (C) $a = 40/35, b = -1/5, c = 2/7$;
   (D) $a = 41/35, b = 0, c = -2/7$;
   (E) $a = 40/35, b = 1/5, c = 1/7$.

7. Which of the following sets consists of the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$?
   (A) $\{3, -4\}$;
   (B) $\{3, 4\}$;
   (C) $\{5, -7\}$;
   (D) $\{4, -3\}$;
   (E) $\{2, -4\}$.

8. Consider two matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ -3 & 4 \end{bmatrix}$. Which of the following statements about the inner product $\langle A, B \rangle$, and their orthogonality is/are true?
   (A) $\langle A,B \rangle = 0$, orthogonal;
   (B) $\langle A, B \rangle = 6$, not orthogonal;
   (C) $\langle A, B \rangle = 1$, not orthogonal;
   (D) $\langle A, B \rangle = 3$, not orthogonal;
   (E) $\langle A,B \rangle = 0$, not orthogonal.

9. Which of the following matrices is the coordinate transformation matrix from a basis of R^2 consisting of $v_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ to another basis of R^2 consisting of $u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?
   (A) $\begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$;
   (B) $\begin{bmatrix} -3 & -4 \\ 4 & -5 \end{bmatrix}$;
   (C) $\begin{bmatrix} 3 & 5 \\ 7 & 8 \end{bmatrix}$;
   (D) $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$;
   (E) $\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$.

10. Which of the following statements is/are true?
   (A) Every nonzero finite-dimensional inner product space has an orthonormal basis.
   (B) Let A be an $m \times n$ matrix with rank n and $m \geq n$. If A^* is the adjoint matrix of A, then A^*A is invertible.
   (C) A periodic function is in an inner product space with infinite linearly independent vectors.
   (D) A square matrix that is diagonalizable must be full ranked.
   (E) Any normal operator in a finite-dimensional inner product space is diagonalizable.

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11. Consider a sample space $S = \{a, b, c, d, e\}$. Which of the following statements is/are true?
    (A) The collection $\mathcal{F} = \{\emptyset, \{a, b\}, \{c, d, e\}, S\}$ of subsets of S is a σ-algebra in S;
    (B) The collection $\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, S\}$ of subsets of S is a σ-algebra in S;
    (C) The collection $\mathcal{F} = \{\emptyset, \{d\}, \{a, c\}, \{b, e\}, \{a, c, d\}, \{b, d, e\}, \{a, b, c, e\}, S\}$ of subsets of S is a σ-algebra in S;
    (D) The assignment $P(\{a, b\}) = 1/2$, $P(\{c, d\}) = 1/4$, $P(\{e\}) = 1/4$ uniquely determines a probability function on the σ-algebra $\mathcal{F} = 2^S$, where 2^S is the power set of S;
    (E) The assignment $P(\{a\}) = 1/2$, $P(\{b\}) = 1/4$, $P(\{c\}) = 1/8$, $P(\{d\}) = 1/8$, $P(\{e\}) = 1/16$ uniquely determines a probability function on the σ-algebra $\mathcal{F} = 2^S$.

12. Consider a random variable X having a cumulative distribution function $F(x)$ and a real-valued function $g(x)$ with

$$F(x) = \begin{cases} 0 & x < -10, \\ 1/8 & -10 \le x < -3, \\ 1/4 & -3 \le x < 1, \\ 3/4 & 1 \le x < 1.5, \\ 7/8 & 1.5 \le x < 4, \\ 1 & 4 \le x \end{cases} \quad \text{and} \quad g(x) = \begin{cases} -6, & x < -2, \\ x - 4, & |x| \le 2, \\ -2, & x > 2. \end{cases}$$

Which of the following statements is/are true?
    (A) X is a discrete random variable with probability mass function $p(-10) = p(-3) = p(1.5) = p(4) = 1/8$, $p(1) = 1/2$, and $p(x) = 0$ for all other $x \in \mathbb{R}$;
    (B) $P(-2 < X < 3) = 3/4$;
    (C) $g(X)$ is a continuous random variable;
    (D) $P(-1.5 \le g(X) < 1.5) = 0$;
    (E) The variance $\mathrm{Var}(g(X))$ of $g(X)$ is 535/256.

13. Let E_1, E_2, E_3, E_4, E_5 be five independent events. Which of the following statements is/are true?
    (A) The complements $E_1^c, E_2^c, E_3^c, E_4^c, E_5^c$ of E_1, E_2, E_3, E_4, E_5 are five independent events;
    (B) E_1^c, E_2, E_3^c, E_5 are four independent events;
    (C) E_2 and $F = E_1 \cap E_3 \cup E_5$ are two independent events;
    (D) $F_1 = E_1 \cup E_5^c$ and $F_2 = (E_3 \cup E_4)^c$ are two independent events;
    (E) E_1, $G_1 = E_2 \cap E_4$, and $G_2 = E_3^c \cup E_5$ are three independent events.

14. Let X be a random variable with the set of possible values $\{-1, 0, 1\}$ and probability mass function $p(-1) = p(0) = p(1) = 1/3$ and $p(x) = 0$ for all other $x \in \mathbb{R}$. Let $Y = X^2$. Which of the following statements is/are true?
    (A) $E(X) = 0$ where $E(X)$ denotes the expectation of X;
    (B) $E(XY) \neq 0$;
    (C) X and Y are uncorrelated;
    (D) X and Y are independent;
    (E) the conditional probabilty $P(X = 0 \mid Y = 1) = 0$.

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15. Let the joint probability mass function of X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{15}(x+y), & \text{if } x = 0,1,2,\ y = 1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following statements is/are true?

(A) Marginal probability mass function $p_X(x) = (2x+3)/15$ if $x = 0,1,2$ and $p_X(x) = 0$ for all other $x \in \mathbb{R}$;

(B) Marginal probability mass function $p_Y(y) = (1+y)/5$ if $y = 1,2$ and $p_Y(y) = 0$ for all other $y \in \mathbb{R}$;

(C) $P(X = 0|Y = 2) = 1/9$;

(D) X and Y are independent;

(E) All the previous statements are not true.

16. Let X and Y be joint Gaussian random variables with zero means and variances σ_X^2 and σ_Y^2, respectively, and their covarinace $\text{Cov}(X,Y) = E(XY)$. Which of the following statements is/are true?

(A) $|\text{Cov}(X,Y)| \le \sigma_X \sigma_Y$;

(B) If $\text{Cov}(X,Y)$=0, then X and Y are independent;

(C) $aX + bY$ is also a zero-mean Gaussian random variable where $a, b \in \mathbb{R}$;

(D) $E[X|Y]$ is also a zero-mean random variable;

(E) $P(|X| > t) \le \sigma_X^2/t^2$ for any $t > 0$.

17. Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} e^{-y}, & \text{if } y > 0, 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Which of the following statements is/are true?

(A) $E(X|Y = 2) = 1/5$;

(B) $E(X|Y = 2) = 1/4$;

(C) $E(X|Y = 2) = 1/3$;

(D) $E(X|Y = 2) = 1/2$;

(E) The conditional variance $\sigma^2_{X|Y=2}$ of X given $Y = 2$ is $1/12$.

18. Let $X_1, X_2, \ldots, X_n$ be independent random variables, and let $M_{X_i}(t)$ be the moment generating function of X_i for $i = 1, 2, \ldots, n$. Let $X = a_1X_1 + a_2X_2 + \cdots + a_nX_n$, where $a_1, a_2, \ldots, a_n$ are in $\mathbb{R}$, and let $M_X(t)$ be the moment generating function of X. Which of the following statements is/are true?

(A) $M_X(t) = a_1M_{X_1}(t) + a_2M_{X_2}(t) + \cdots + a_nM_{X_n}(t)$;

(B) $M_X(t) = (a_1M_{X_1}(t)) \cdot (a_2M_{X_2}(t)) \cdots (a_nM_{X_n}(t))$;

(C) $M_X(t) = M_{X_1}(a_1t) \cdot M_{X_2}(a_2t) \cdots M_{X_n}(a_nt)$;

(D) $M_X(t) = (M_{X_1}(t))^{a_1} + (M_{X_2}(t))^{a_2} + \cdots + (M_{X_n}(t))^{a_n}$;

(E) $M_X(t) = (M_{X_1}(t))^{a_1} \cdot (M_{X_2}(t))^{a_2} \cdots (M_{X_n}(t))^{a_n}$.

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19. Let X be a Gaussian random variable with mean μ and variance σ^2, i.e., $X \sim N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma^2 > 0$. It is known that the moment generating function $M_X(t)$ of X is given by $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ for $t \in R$. Let $Y = e^X$. Which of the following statements is/are true?

(A) $E[Y] = e^{\mu+\sigma^2}$;

(B) $E[Y^2] = e^{2\mu+2\sigma^2}$;

(C) $Var(Y) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$;

(D) $E[Y^3] = e^{3\mu+3\sigma^2}$;

(E) $E[Y^4] = e^{4\mu+4\sigma^2}$.

20. Let X be a discrete random variable and let the probability mass function $p(x)$ of X be given by $p(1) = 0.4$, $p(2) = 0.3$, $p(3) = 0.2$, $p(4) = 0.1$, and $p(x) = 0$ for all other $x \in \mathbb{R}$. The Markov inequality says that $P(X \geq t) \leq \frac{E[X]}{t}$ for $t > 0$, and the Chebyshev inequality says that $P(|X - E[X]| \geq t) \leq \frac{Var(X)}{t^2}$ for $t > 0$. Which of the following statements is/are true?

(A) By using the Markov inequality, we obtain $P(X \geq 4) \leq 0.25$;

(B) By using the Chebyshev inequality, we obtain $P(X \geq 4) \leq 0.25$;

(C) The Markov inequality gives a better upper bound for $P(X \geq 4)$ than the Chebyshev inequality;

(D) The Chebyshev inequality gives a better upper bound for $P(X \geq 4)$ than the Markov inequality;

(E) The Markov inequality and the Chebyshev inequality give the same upper bound for $P(X \geq 4)$.