＊本科考試可使用計算器，廠牌、功能不拘

1. (計算題，20分)

Adopt the following procedure to calculate the inverse of matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} -3 & -0.5 & 1 & 1.5 \\ 1 & 0.25 & -0.5 & -0.25 \\ 3 & 0.25 & -0.5 & -1.25 \\ -3 & 0 & 1 & 1 \end{bmatrix}.$$

(a) Set up an augmented matrix $[\mathbf{A}|\mathbf{I}]$, where $\mathbf{I}$ is the 4×4 identity matrix.

(b) Apply "row operations" (e.g., row swaps, scaling rows, and adding multiples of one row to another row) to transform the augmented matrix $[\mathbf{A}|\mathbf{I}]$ into the form $[\mathbf{I}|\mathbf{B}]$.

(c) If you can successfully transform $[\mathbf{A}|\mathbf{I}]$ into $[\mathbf{I}|\mathbf{B}]$, then matrix $\mathbf{B}$ is the inverse of matrix $\mathbf{A}$.

Note: You must show step-by-step calculation processes to receive a full score. Moreover, if you do not use the above procedure, you will not receive any score.

2. (計算題，30分)

Solve the following optimization problem using the "dual simplex" method.

$$\begin{aligned} \text{Minimize} \quad & 4x_1 + 6x_2 + 8x_3 \\ \text{subject to} \quad & 4x_1 - 2x_2 + 6x_3 \geq 8 \\ & 3x_1 + 6x_2 + 3x_3 \geq 9 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Note: You have to show detailed calculation processes to receive a full score. Moreover, if you do not use the "dual simplex" method, you will not receive any score.

3. (建立模型，4個子題，共50分)

【背景描述】

John is a contract logistics professional working for a major unmanned aerial vehicle (UAV) maker in the country. His work is to collect high-value parts made by various electronic component makers in different cities and then transport these electronic parts to the UAV maker. The area in which these component makers (cities) are located is big. Therefore, due to long-distance travel, work regulations and other reasons, John has made an agreement with the UAV maker that he would just make one delivery trip for them every day except weekend and holidays; moreover, due to his long working relationship with them, he gets to choose any city that he likes to maximize income (explained shortly). Typically, John would drive his pickup truck leaving home around 7 AM; then, he would visit some or all of the component makers to collect finished parts and transport them to the UAV maker at or before 5 PM on the same day. John's home and the UAV maker are in the same city.

Being a contract logistics professional, John is paid on a trip-basis such that his pay is precisely 2.0% of the total worth of the electronic parts that he is able to deliver to the UAV maker at the end of his daily trip. As an example, if one day John delivered a total worth of 500,000 NT (台幣) of electronic parts to the UAV maker, he would get 10,000 NT for doing the work. But John is responsible for all the fuel costs that are required to finish the trip. Therefore, if he paid 2,500 NT for the fuel of his truck that day, his actual income is just 7,500 NT. As every component maker has a production schedule different from all other makers, the quantity of finished parts that John can collect from each maker is different every day. So, it is very important for John to determine his travel route carefully before leaving home every morning, as which city to go collecting finished parts and how to move among them determines the actual income that he can make that day!

【模型開發】

The above situation can be solved optimally by using a mixed-integer programming (MIP) model. To make the problem easier, let's assume that all the cities (including the one in which John and the UAV maker reside) form a "complete graph" such that each city is a "node" in the graph and any two cities are connected by a "link." Obviously, this link provides a direct way to travel from one city to another, without the need to go to a third city first. Let's number all the cities with 1, 2, 3, etc., with "1" being the city in which John and the UAV maker are located. The following figure depicts an example of a complete graph with six cities.

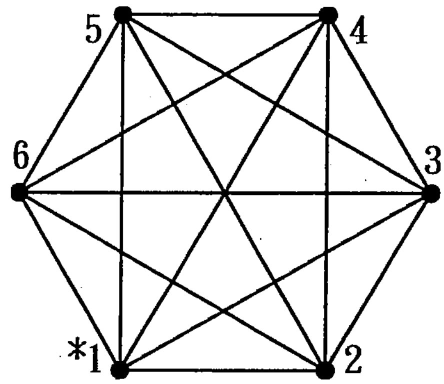

Figure 1. A complete graph with 6 cities.

The following describes the finished electronic parts that John can collect in the complete graph.

- Every node except node 1 has certain finished electronic parts that John can collect. And he knows their total worth and total weight before leaving home at node 1 in the morning.

The following formally describes the requirements for John's pickup-and-delivery trip.

- John will start his trip from node 1.
- Before returning to node 1, John may visit any other node in the graph (note: it is "may," not "must"). And no matter how he travels, he cannot visit any node more than once.
- Whenever he visits a new node after leaving node 1, he must pick up all the finished parts sitting over there (that is, he cannot just "pass by" a node and do nothing about the finished parts that he can pick up over there; partial pickup is NOT allowed, either).
- There is a maximum load limit for the pickup truck. Therefore, John must choose the nodes to visit carefully to make sure that their finished parts have a total weight less than or equal to the truck's maximum load limit.

Here is an example for the kind of trip that John can make: 1→3→5→1. In this example, he visits nodes 3 and 5 exactly once after leaving node 1, picks up all the finished parts at those two nodes (assuming the parts' total weight is OK for the truck) and then return to node 1. We call 1→3→5→1 a node sequence (節點序列), as it is a sequence of nodes that John can use to do his work.

The following list contains some of the parameters and indices related to developing a MIP model for John. Note that “parameters” mean known information or given data (給定資訊); in other words, they are NOT decision variables.

N	parameter, indicating the set containing all the nodes at which high-value UAV parts are made and node 1 at which John and the UAV maker are located
n	index, representing one of the cities in **N** (that is, $n \in \mathbf{N}$)
$\mathbf{V}_n$	parameter, indicating the total worth of the parts that John can pick up at node n (for example, $\mathbf{V}_5 = 150{,}000$ if the total worth of the parts at node 5 is 150,000 NT)
$\mathbf{W}_n$	parameter, indicating the total weight of the parts that John can pick up at node n (for example, $\mathbf{W}_{13} = 250$ if the total weight of the parts at node 13 is 250 kg)
$\mathbf{W}^{\text{truck}}$	parameter, indicating the maximum load limit for John’s pickup truck (for example, $\mathbf{W}^{\text{truck}}$ could be 3,000 kg)
$\mathbf{F}_{i,j}$	parameter, indicating the fuel costs required for traveling from node i to node j or vice versa (for example, $\mathbf{F}_{7,18}$ could be 450 NT)

The following are four steps to guide you to develop a MIP model to select **an optimal node sequence for John**, that is, if he visits and picks up finished parts according to that node sequence, his actual income will be maximized.

Step 1 (20 分):

To develop such a MIP model, one possible approach is to construct “a set of node sequences” and then use them to calculate certain parameters for the model (先建構一些既定的節點序列，然後再用這些節點序列來計算一些模型需要的參數). Let **S** be such a node-sequence set. Please describe in detail as to the kind of node sequences that you will want **S** to contain. (令S為一個包含節點序列的集合。請問這個集合應該包含什麼樣的節點序列，才能有助於模型求得最佳解。請清楚說明。你可以用一個簡單的例子，使說明更加清楚。)

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Step 2 (10 分):

With the determination of node sequences that $\mathbf{S}$ should contain in Step 1, now you can calculate three parameters for each node sequence $s \in \mathbf{S}$ (你可以為集合S裡面的每一條節點序列計算三種參數):

1. the total worth of electronic parts that John must pick up from every component-maker node contained in s (denoted as $\mathbf{V}_s^{\text{sequence}}$)
2. the total weight of those parts (denoted as $\mathbf{W}_s^{\text{sequence}}$)
3. the total fuel costs for John to travel following node sequence s (denoted as $\mathbf{F}_s^{\text{sequence}}$).

Please elaborate in detail as to how you will calculate $\mathbf{V}_s^{\text{sequence}}$, $\mathbf{W}_s^{\text{sequence}}$ and $\mathbf{F}_s^{\text{sequence}}$. (請說明你會如何去計算這三種參數)

Now we are ready to develop the MIP model. Let x_s be a <u>binary</u> decision variable such that $x_s = 1$ if node sequence s is selected for John to travel, and $x_s = 0$ if s is not selected for him.

Step 3 (10 分):

The MIP model has an objective function to maximize the actual income for John. What is its mathematical expression? (請利用上面給定的資訊，寫出模型的目標式)

Step 4 (10 分):

The MIP model only needs <u>two</u> very simple constraints to be totally complete. What are these two constraints? (模型只需要兩條非常簡單的限制式；請利用上面給定的資訊，將它們寫出來)