- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請將答案用2B鉛筆填於答案卡。
- 共二十題，每題五分。每題ABCDE選項單獨計分；每一選項個別分數為一分，答錯倒扣一分，倒扣至本測驗試題零分為止。

Notation: In the following questions, $\mathbb{Z}$ is the set of integers, and $\mathbb{R}$ is the usual set of all real numbers. We will use underlined lowercase letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n. $\|\underline{a}\|$ means the Frobenius norm of vector $\underline{a}$, and $\underline{0}$ is the all-zero column vector of proper length. We will use boldface uppercase letters such as $\mathbf{A} \in \mathbb{R}^{m\times n}$ to denote a real matrix $\mathbf{A}$ of size $m \times n$, and we will write $\mathbf{A} = [a_{i,j}] \in \mathbb{R}^{m\times n}$, where $a_{i,j}$ is the (i, j)-th entry of $\mathbf{A}$ with subindices $i = 1, \dots, m$, and $j = 1, \dots, n$. $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. $\det(\mathbf{A})$ and $\operatorname{tr}(\mathbf{A})$ are respectively the determinant and trace of square matrix $\mathbf{A}$. $\operatorname{row}(\mathbf{A})$, $\operatorname{col}(\mathbf{A})$ and $\operatorname{null}(\mathbf{A})$ are the row, column and right null spaces of $\mathbf{A}$ over $\mathbb{R}$, respectively. Unless otherwise stated, all vector spaces and linear combinations are over field $\mathbb{R}$, and the orthogonality is with respect to the usual Euclidean inner product. By $\dim(\mathcal{V})$ we mean the dimension of vector space $\mathcal{V}$ over its base field $\mathbb{R}$.

We will write $F_X(x)$ to denote the cumulative distribution function (CDF) of random variable X. If X is a discrete random variable, then the probability mass function (PMF) of X is denoted by $p_X(x)$; if X is a continuous random variable, it is always assumed that X has a probability density function (PDF), denoted by $f_X(x)$. $\mathbb{E}[X]$ and $\operatorname{Var}(X)$ denote the expected value and variance of random variable X, respectively. $\mathbb{P}()$ denotes the probability measure in a probability space.

1. Consider the linear system $\mathbf{A}\underline{x} = \underline{b}$ in the unknown $\underline{x}$, where $\mathbf{A} = [\underline{a}_1, \dots, \underline{a}_n] = [\underline{r}_1, \dots, \underline{r}_m]^\top \in \mathbb{R}^{m\times n}$ and nonzero $\underline{b} = [b_1, \dots, b_m]^\top \in \mathbb{R}^m$ are given and fixed, with $\underline{r}_i^\top$ and $\underline{a}_j$ representing the i-th row and the j-th column of matrix $\mathbf{A}$, respectively. Which of the following statements is/are true?

    (A) The number of pivots of $\mathbf{A}$ equals the minimum of m and n.

    (B) With $\underline{a}_j$'s and $\underline{b}$ given above, if the linear system $\sum_{j=1}^n y_j \underline{a}_j = \frac{1}{2}\underline{b}$ in the unknowns y_j's is consistent, then $\underline{b} \in \operatorname{col}(\mathbf{A})$.

    (C) Let $\underline{s}$ be a solution to the above system $\mathbf{A}\underline{x} = \underline{b}$. If $\underline{r}_i^\top \underline{s} = 0$ for some i, then $\underline{r}_k^\top \underline{s} = 0$ for all $k = 1, \dots, m$.

    (D) If the system $\mathbf{A}\underline{x} = \underline{b}$ has infinitely many solutions, then there exists a vector $\underline{x}_0 \in \mathbb{R}^n$ with $\underline{x}_0 \neq \underline{0}$ such that $\mathbf{A}\underline{x}_0 = \underline{0}$.

    (E) None of the above is true.

注意：背面有試題

2. Consider the three linear transformations $H_i : \mathbb{R}^2 \to \mathbb{R}^2$ given by $H_i(\underline{s}) = \mathbf{H}_i\,\underline{s} = \underline{r}_i$ for $i = 1, 2, 3$, where

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{H}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{H}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

respectively. Which of the following statements is/are true?

(A) For every $\underline{s}$, the corresponding $\underline{r}_i$ is in col($\mathbf{H}_i$) for all $i = 1, 2, 3$.

(B) For every $\underline{s}$, the corresponding $\underline{r}_i$ is in null($\mathbf{H}_i$) for all $i = 1, 2, 3$.

(C) There exists $\underline{s}$ such that the corresponding $\underline{r}_i^\top$ is in row($\mathbf{H}_i$) for all $i = 1, 2, 3$.

(D) There exists a non-zero vector in the intersection of col($\mathbf{H}_1$), col($\mathbf{H}_2$) and col($\mathbf{H}_3$).

(E) None of the above is true.

3. Continue from Question 2. Which of the following statements is/are true?

(A) There exists $\underline{s}$ such that the corresponding $\underline{r}_1 = \underline{r}_2 \neq \underline{0}$.

(B) There exists $\underline{s}$ such that the corresponding $\underline{r}_1 \neq \underline{r}_2 = \underline{0}$.

(C) There exists $\underline{s}$ such that the corresponding $\underline{r}_2 \neq \underline{r}_3 = \underline{0}$.

(D) There exists $\underline{s}$ such that the corresponding $\underline{r}_1 \neq \underline{r}_2 = \underline{r}_3 = \underline{0}$.

(E) None of the above is true.

4. The following set is a basis for the vector space $\mathbb{R}^3$

$$\mathcal{B} = \left\{ \underline{b}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \underline{b}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

It means that every vector in $\mathbb{R}^3$ can be expressed as a linear combination of the three basis vectors, $\underline{b}_1$, $\underline{b}_2$ and $\underline{b}_3$. For instance, $\underline{y} = [1, 2, 3]^\top = \sum_{i=1}^3 a_i \underline{b}_i$ for some a_1, a_2 and a_3, and the vector $\underline{a} = [a_1, a_2, a_3]^\top$ is called the coordinate vector of $\underline{y}$ with respect to basis $\mathcal{B}$. Which of the following statements is/are true?

(A) The basis $\mathcal{B}$ is NOT an orthogonal basis.

(B) The basis $\mathcal{B}$ can be obtained by a rotation of the standard basis for $\mathbb{R}^3$.

(C) $4 \leq \|\underline{a}\| \leq 5$.

(D) $3 \leq \|\underline{a}\| \leq 4$.

(E) None of the above is true.

5. Continue from Question 4. Let

$$\mathcal{C} = \left\{ \underline{c}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \underline{c}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \underline{c}_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \right\}$$

be another basis for $\mathbb{R}^3$. The coordinate vector of $\underline{y}$ with respect to $\mathcal{C}$ is $\underline{a}' = [a'_1, a'_2, a'_3]^\top$, and we have $\underline{a}' = \mathbf{W}\underline{a}$ for some matrix $\mathbf{W}$. Which of the following statements is/are true?

(A) $\mathbf{W}$ is a square matrix.

(B) $\det(\mathbf{W}) > 0$.

(C) $\underline{a}'$ is orthogonal to $\underline{a}$.

(D) $\|\underline{a}'\| < \|\underline{a}\|$.

(E) None of the above is true.

注意：背面有試題

6. Consider the linear transformation $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ given by

$$T(\mathbf{A}) = \mathbf{AM} - \mathbf{MA}, \text{ where } \mathbf{M} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Let $\mathbf{T}$ be the matrix representation of T with respect to the following basis $\mathcal{B}$ for $\mathbb{R}^{2\times2}$

$$\mathcal{B} = \left\{ \mathbf{B}_1 = \begin{bmatrix} 2 & -3 \\ 3 & 3 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 1 & -2 \\ -3 & 0 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 3 & -2 \\ 3 & 3 \end{bmatrix}, \mathbf{B}_4 = \begin{bmatrix} 3 & 2 \\ 0 & -3 \end{bmatrix} \right\}.$$

Which of the following statements is/are true?

(A) $\mathbf{T}$ is symmetric and nonnegative definite.

(B) The absolute of the product of nonzero singular values of $\mathbf{T}$ equals 16.

(C) The maximal eigenvalue of $\mathbf{T}$ exceeds 5.

(D) $\dim(\text{row}(\mathbf{T})) = 3$.

(E) None of the above is true.

7. Continue from Question 6. Let $\mathbf{C} = [c_{i,j}]$ be the orthogonal projection of $\mathbf{B}_1$ onto the kernel space of T with respect to the inner product $\langle \mathbf{A}, \mathbf{B} \rangle := \text{tr}(\mathbf{A}^\top \mathbf{B}) + \text{tr}(\mathbf{B}^\top \mathbf{A})$ for any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times2}$. Which of the following statement is/are true?

(A) $c_{1,1} > 2$.

(B) $c_{1,2} < -3$.

(C) $c_{2,1} > 3$.

(D) $c_{2,2} < 3$.

(E) None of the above is true.

注意：背面有試題

8. Continue from Question 6. Let $\mathcal{V}$ be the vector space generated by compositions of T over $\mathbb{R}$, i.e.,

$$\mathcal{V} = \left\{ \sum_{n=0}^{\infty} v_n T^n : v_n \in \mathbb{R} \right\},$$

where $T^0(\mathbf{A}) = \mathbf{A}$ is the identity map and by T^n we mean the n-fold composition of T, and let $G : \mathcal{V} \to \mathcal{V}$ be a linear operator given by $G(v) = T^2 v$ for $v \in \mathcal{V}$. Which of the following statement is/are true?

(A) $\dim(\mathcal{V}) = 4$.

(B) The kernel space of G is trivial.

(C) The sum of all eigenvalues of G exceeds 4.

(D) G has four linearly independent eigenvectors.

(E) None of the above is true.

9. Let $\mathcal{P}$ be the vector space of real polynomials $p(x)$ having a common zero at $x = 1$. The space $\mathcal{P}$ is equipped with an inner product $\langle p(x), q(x) \rangle := \sum_{i,j \geq 0} \frac{p_i q_j}{i+j+1}$ for any $p(x) = \sum_{i \geq 0} p_i x^i$ and $q(x) = \sum_{j \geq 0} q_j x^j$ in $\mathcal{P}$. Let $q(x) = q_3 x^3 + q_2 x^2 + q_1 x + q_0$ be the orthogonal projection of $x^3 - x^2 + 2x - 2$ onto the orthogonal complement of the subspace of linear polynomials in $\mathcal{P}$. Which of the following statement is/are true?

(A) $q_2 > q_1$.

(B) $q_2 > q_0$.

(C) $q_1 > q_0$.

(D) $q_0 > 0$.

(E) None of the above is true.

注意：背面有試題

10. Consider the QR-decomposition of the following matrix

$$\begin{bmatrix} 2 & 0 & 3 \\ -2 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \mathbf{QR},$$

where the columns of orthonormal matrix $\mathbf{Q} = [q_{i,j}]$ are obtained using the usual Gram-Schmidt procedure but are chosen such that the diagonal values of upper triangular matrix $\mathbf{R} = [r_{i,j}]$ are all negative. Which of the following statement is/are true?

(A) $q_{1,2} > 0$.

(B) $q_{2,2} > 0$.

(C) $r_{2,3} > 0$.

(D) $r_{1,2} > r_{1,3}$.

(E) None of the above is true.

11. Let $X \in \{0, 1\}$ be a Bernoulli random variable with $\mathbb{E}[X] = \frac{1}{4}$. Let Y, $0 \leq Y \in \mathbb{Z}$, be a Poisson random variable with $\mathbb{E}[Y] = 1$. In addition, X and Y are statistically independent. Which of the following statements is/are true?

(A) $\mathbb{P}(Y = 1) = e^{-1}$.

(B) $\mathbb{P}(Y = 2) = e^{-2}$.

(C) $\mathbb{P}(X + Y = 0) \leq \frac{3}{8}$.

(D) $\mathbb{P}(2X + Y = 2) = \frac{5}{8}e^{-1}$.

(E) $\mathbb{P}(X > Y) = e^{-1}$.

注意：背面有試題

12. Let $X \in \mathbb{R}$ be a continuous random variable with the following CDF

$$F_X(x) = \begin{cases} \frac{1}{2}e^{x}, & x \leq 0, \\ 1 - \frac{1}{2}e^{-x}, & x > 0. \end{cases}$$

Which of the following statements is/are true?

(A) $F_X(-4) \leq F_X(4)$.

(B) $\mathbb{P}(X \leq 2) = 1 - \frac{1}{2}e^{-1}$.

(C) $f_X(x) \geq 0$ for all $x \in \mathbb{R}$.

(D) $f_X(-2) = f_X(2)$.

(E) $f_X(x)F_X(x) \in [0, 1]$ for all $x \in \mathbb{R}$.

13. Continue from Question 12. Which of the following statements is/are true?

(A) $\mathbb{E}[X] = 0$.

(B) $\mathbb{P}(X \in [0, 1]) \geq \frac{1}{3}$.

(C) $\mathbb{E}[X^3] = 1$.

(D) $\mathbb{E}[X^2] + \mathbb{E}[X^4] = 25$.

(E) $\mathbb{E}[X^2] = 2$.

注意：背面有試題

14. Let X be a continuous random variable that is uniformly distributed in $[1,2]$. Let $Y = X^2$. Which of the following statements is/are true?

(A) $\mathbb{P}(Y \geq 2) \geq \frac{1}{2}$.

(B) $\mathbb{P}(Y \in [2,3]) = \frac{1}{3}$.

(C) $\mathbb{P}(Y = 2) \geq \frac{1}{4}$.

(D) $\mathbb{P}(Y \leq 3) = \sqrt{3} - 1$.

(E) $f_Y(2) \leq \frac{1}{4}$.

15. Let X and Y be two independent real Gaussian random variables with $\mathbb{E}[X] = 2$, $\mathbb{E}[X^2] = 5$, $\mathbb{E}[Y] = 0$ and $\mathbb{E}[Y^2] = 1$. It can be shown that $\mathbb{E}[-2\ln(f_X(X))] = \ln(2^a e^b \pi^c)$ for some $a, b, c \in \mathbb{Z}$. Which of the following statements is/are true?

(A) $\mathbb{P}(X \geq 3) = \frac{1}{2}$.

(B) $\mathbb{E}[(X-2)^4] = 3$.

(C) $\mathrm{Var}(X+Y) = 2$.

(D) $a = 1$ and $b = -1$.

(E) $c = 1$.

16. Consider a Bernoulli random variable $X \in \{0,1\}$ with $\mathbb{P}(X=1)=q$ and $\mathbb{P}(X=0)=1-q$, and a continuous random variable Y that is conditioned on X. Given $X=1$, Y is a real Gaussian random variable with mean μ and variance σ^2; on the flip side, given $X=0$, Y is an exponential random variable with mean λ. Which of the following statements is/are true?

(A) The joint PDF $f_{X,Y}(1,3)=\frac{1}{\lambda}e^{-\frac{3}{\lambda}}$.

(B) The mean of Y is $\lambda+q(\mu-\lambda)$.

(C) $\mathbb{E}[Y^2]=8$ when $\mu=-2$, $\lambda=2$ and $\sigma=2$.

(D) $\mathbb{E}[Y^2]>10$ when $\mu=-3$, $\lambda=2$ and $\sigma=1$.

(E) None of the above is true.

17. Continue from Question 16. Which of the following statements is/are true when $q=\frac{1}{2}$?

(A) $f_Y(y)=\frac{1}{\sqrt{8\pi\sigma^2}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$ for $y<0$.

(B) The moment generation function of Y is $M_Y(s)=e^{\mu s+\frac{\sigma^2 s^2}{2}}$.

(C) $\mathrm{Var}(Y)>6$ when $\mu=1$, $\lambda=2$ and $\sigma=3$.

(D) $\mathrm{Var}(Y)<2$ when $\mu=-1$ and $\lambda=\sigma=1$.

(E) None of the above is true.

18. Let $X_i \in \{0,1\}$ with $i = 1, \ldots, N$, be independent and identically distributed Bernoulli random variables with $\mathbb{E}[X_i] = p$, where N is a Poisson random variable, independent of X_i, with mean λ. Let $L = X_1 + \ldots + X_N$ and $M_L(s)$ be the moment generation function of L. Which of the following statements is/are true?

(A) $\mathbb{E}[L] = \frac{p}{\lambda}$.

(B) $\mathrm{Var}(L) = \lambda p$.

(C) $M_L(-\ln(2)) = e^{-2\lambda p}$.

(D) $M_L(\ln(2)) = e^{\lambda p}$.

(E) None of the above is true.

19. Which of following statements is/are true?

(A) If X is an exponential random variable with mean equal to 1, then $\mathbb{P}(X \geq a) \leq \frac{1}{a}$ for all real $a > 0$.

(B) If Y is a real Gaussian random variable with zero mean and unit variance, then $\mathbb{P}(Y^2 + 2Y + 1 \geq a) \leq \frac{3}{a}$ for all real $a > 0$.

(C) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_i] = 5$ for $i = 1, 2, \ldots$; then $\lim_{n\to\infty} \mathbb{P}(\left|5 - \frac{1}{2n}\sum_{i=1}^{n} X_i\right| \geq 0.01) = 0$.

(D) Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed Poisson random variables with $\mathrm{Var}(Y_i) = 4$ for $i = 1, 2, \ldots$, and let $\bar{Y}_n = \frac{\sqrt{n}}{2}\left(4 - \frac{1}{n}\sum_{i=1}^{n} Y_i\right)$; then $\lim_{n\to\infty} \mathbb{P}(\bar{Y}_n \leq 1) > \frac{1}{2}$.

(E) None of the above is true.

20. Let X and Y be random variables with the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2}, & \text{if } x^2+y^2 \leq r^2, \\ 0, & \text{if otherwise,} \end{cases}$$

for all $x, y \in \mathbb{R}$ and for some fixed real $r > 0$. Which of following statements is/are true?

(A) $\mathbb{E}\left[X \mid Y = \frac{r}{2}\right] = \frac{1}{2}$.

(B) $\mathbb{E}\left[Y \mid X = \frac{r}{3}\right] = 0$.

(C) X and Y are statistically dependent.

(D) X and Y are statistically correlated.

(E) None of the above is true.