1. Differential Equations:

[10%] (a) Solve the ordinary differential equation: $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2 + 2$

[10%] (b) Determine the general solution of the following partial differential equation,

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = z$$

2. Vector Analysis:

[6%] (a) Show that the area enclosed by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to πab.

[4%] (b) Determine a unit vector normal to the surface $x^3 - xyz + z^3 = 5$ at point (1,1,1) in Cartesian coordinates.

[10%] (c) Determine the unit tangent vector and the radius of curvature of a right circular helix:

$$x = a\cos t, \quad y = a\sin t, \quad z = ct$$

where a and c are non-zero constants.

3. Consider a Pauli spin martix $\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

[6%] (a) Find its eigen values λ_1, λ_2, and normalized eigen vectors, $\hat{e}_1$, $\hat{e}_2$, where $\lambda_1 > \lambda_2$.

[4%] (b) Find the transformation matrix S, such that $S\sigma S^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $SS^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

[10%] 4. The operator $\nabla^2 - \frac{\partial^2}{\partial t^2}$ in 4 dimensional space-time is called the d'Alembertian. Show that it is a scalar operator.

5. Let the improper integral $I(a) = \int_{-\infty}^{+\infty} \frac{dx}{x^2 - a^2}$, where a is real and positive.

[10%] (a) Find $I(a)$ by taking the Cauch principal value.

[5%] (b) Find $I(a)$ by taking $I(a) = \lim_{\gamma\to 0} I(a + i\gamma)$

[5%] (c) Find $I(a)$ by taking $I(a) = \lim_{\gamma\to 0} I(a - i\gamma)$

where γ is positive and real.

[20%] 6. The homogeneous Helmhotz equation in three dimensional space is

$$\nabla^2\Phi + k^2\Phi = 0$$

(a) Find eigen functions of the equation.

(b) Expand the Green's function of ∇^2 in terms of the eigen functions