國立中央大學九十一學年度碩士班研究生入學試題卷

所別：地球物理研究所　不分組　科目：微積分　共 2 頁 第 1 頁
應用地質研究所　不分組

1. Briefly explain the following in the limit of 100 words:

(i) What is the Fourier Analysis and write an example of its application? (5%)

(ii) What is the Calculus of Variations and write an example of its application? (5%)

(iii) In the Probability Theory, what is the relation between probability function and distribution function? (5%)

2. The Earth has the shape of a slightly flattened sphere, i.e. the polar radius is slightly less than the equatorial radius. Figure 1 is an exaggeration of this to illustrate the point. The precise shape is one in which sections cut parallel to the equator are circular whilst sections through the poles are elliptical. Such a shape is called an ellipsoid. In an ellipsoid, the radius r of a circle of constant latitude at a distance z from the equator is given by

$$r^2 = r_e^2[1-(z^2/r_p^2)] \qquad \text{(Eq. 1)}$$

(i) Write an integral expression for the volume of the Earth by assuming it is filled with an infinite number of infinitely thin discs (i.e. $\Delta z \to 0$ in Fig. 1), and evaluate the resulting integral. (5%)

(ii) If the equatorial radius of the Earth is 6378 km and the polar radius is 6357 km, what is the Earth's volume? (5%)

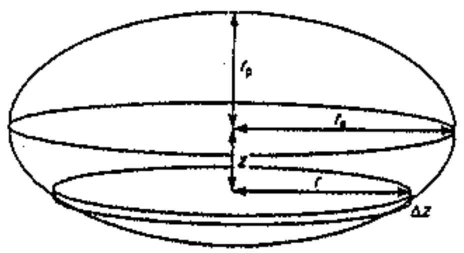

Figure 1 The Earth is an ellipsoidal shape whose equatorial radius r_e is slightly greater than its polar radius r_p. A thin disc is shown which is parallel to the equator, has a radius r and thickness Δz and is a vertical distance z from the equator.

3. If (x_1,t_1), (x_2,t_2), …, (x_n,t_n) represent our n observed arrival times of seismic waves at n seismic stations at distances x_i ($i = 1, 2, \ldots, n$), and suppose that the relationship between t and x is given by the following linear equation:

$$t = a\,x + b \qquad \text{(Eq. 2)}$$

where a is the inverse of wave propagating velocity and b is related to the thickness of the layer. The Least-Square method is a simple technique for finding a "best-fit" straight line that passes very close to all of the observed points, i.e. minimizes the sum of the squares of the differences between the observed t_i and its prediction.

(i) Please derive the formula of the Least Square method for a and b by the minimum concept in calculus. (10%)

(ii) Again, please derive the formula for a and b but using the matrix formulation this time. (10%) (Hint: The inverse matrix of an arbitrary 2x2 matrix $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is $\dfrac{1}{m_{11}m_{22}-m_{12}m_{21}}\begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$.)

4. A damping motion of a unit mass on a spring, $y(t)$, is governed by a second-order differential equation:

$$\ddot{y} + c\dot{y} + ky = 0 \qquad \text{(Eq. 3)}$$

where c is the damping factor and k is the spring constant, and both are constants.

(i) Please convert (Eq. 3) to a set of two first-order differential equations, i.e. a first-order differential system. (5%) (Hint: By setting $y_1 = y$ and $y_2 = \dot{y}$)

(ii) Find the eigenvalues of the coefficient matrix for your linear system in (i). (5%)

(iii) Assuming that $c = 2$ and $k = 0.75$, find again the eigenvalues and their corresponding eigenvectors, and then write the general solution for $y(t)$. (10%)

5. Solve the boundary-value problem:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \qquad u(0, y) = 8e^{-3y} \qquad \text{(Eq. 4)}$$

by the method of separation of variables. Here u is a function of x and y. (10%)

6. Considering a one-dimensional motion of a particle is governed by the following nonlinear differential equation:

$$\dot{x} = \sin x \qquad \text{(Eq. 5)}$$

where x is the position of this particle and $\dot{x}$ is the time-derivative of its position, i.e. the velocity.

(i) Suppose that $x(t=0) = x_0$, please find the position function $x(t)$ for this particle. (5%) (Hint: $\int \csc u\, du = -\ln|\csc u + \cot u| + C$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

(ii) Assuming $x_0 = \pi/4$, please plot the solution $x(t)$ qualitatively. What happens as $t \to \infty$? (5%)

(iii) Now let consider another thinking way for (Eq. 5). Please plot (Eq. 5) in the so-called phase plane, that is the position as the abscissa and the velocity as the ordinate or the x-$\dot{x}$ plane, and simply by arrows indicate the moving directions of our imaginary particle everywhere. (5%)

(iv) For an arbitrary initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$? (5%) (Hint: Consider those points $x = n\pi$, n is an integer)

(v) For any one-dimensional system $\dot{x} = f(x)$ as shown in Fig. 2, could you describe the general behavior of the particle motion qualitatively as the evolution of time? (5%)

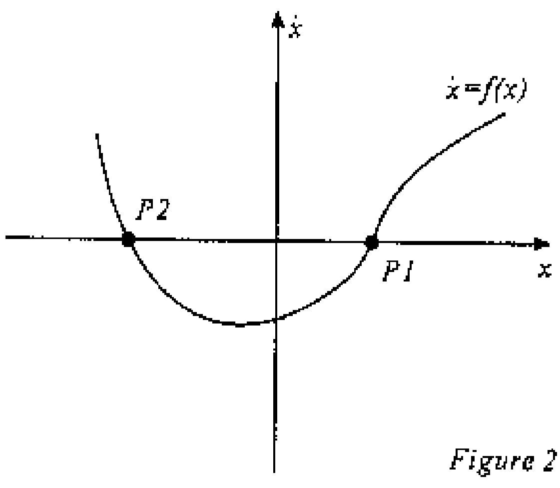

Figure 2