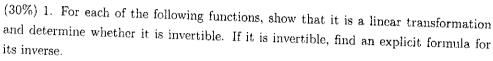
科目: 線性代數(1002)

校系所組:中大數學系甲組、乙組

## 清大數學系純粹數學組、應用數學組



(15%) (a)  $T: P_3(\mathbb{R}) \to P_3(\mathbb{R})$  defined by  $T(f(x)) = \int_x^{x+1} f(t)dt$ , where  $P_3(\mathbb{R})$  denotes the space consisting of polynomials with real coefficients of degree less than or equal to 3.

(15%) (b)  $T: M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$  defined by  $T(A) = A + \operatorname{tr}(A)I_n$ , where  $M_{n\times n}(\mathbb{R})$  denotes the space consisting of  $n\times n$  matrices with real entries,  $\operatorname{tr}(A)$  denotes the trace of A, and  $I_n$  denotes the identity matrix.

(15%) 2. Let V be a finite dimensional inner product space over the real numbers and let  $T: V \to V$  be a linear operator on V. Define  $\det(T) = \det([T]_{\beta})$ , where  $[T]_{\beta}$  denotes the matrix representation of T in the ordered basis  $\beta$ .

(5%) (a) Show that det(T) is independent of the choice of the ordered basis  $\beta$ .

(10%) (b) Suppose that ||T(v)|| = ||v|| for all  $v \in V$ . Prove that  $\det(T) = \pm 1$ .

(10%) 3. Let V be the space of polynomials with real coefficients with inner product  $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$ . Let W be the subspace consisting of even polynomials. Prove or disprove that  $V = W \oplus W^{\perp}$ , where  $W^{\perp}$  is the orthogonal complement of W.

(15%) 4. For  $B \in M_{m \times m}(\mathbb{R})$ , define the linear operator T on the matrix space  $M_{m \times n}(\mathbb{R})$  by T(A) = BA.

(5%) (a) Prove that T is is invertible if and only if B is invertible.

(5%) (b) Prove that T and B have the same eigenvalues.

(5%) (c) Prove that T is diagonalizable if and only if B is diagonalizable.

(15%) 5. Let r be a positive real number and let m be a positive integer. Let  $A = [a_{ij}]$  be an  $m \times m$  matrix given by

$$a_{ij} = \begin{cases} r^{i-1} & \text{if } i+j=1+m, \\ 0 & \text{otherwise.} \end{cases}$$

(5%) (a) Show that  $\pm (\sqrt{r})^{m-1}$  are the only eigenvalues of A.

(10%) (b) Find the minimal polynomial of A and evaluate  $A^{2008}$ .

(15%) 6. Let V be a finite dimensional vector space over a field F and let  $T: V \to V$  be a nonzero linear operator on V. Denote the minimal polynomial of T by f(x). Prove or disprove that  $V = N(T) \oplus R(T)$  if and only if  $x^2 \nmid f(x)$ .

