- 本測驗試題為多選題(答案可能有一個或多個)，請選出所有正確或最適當的答案，並請將答案用2B鉛筆填於答案卡。
- 共二十題，每題五分。每題ABCDE選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分，倒扣至本測驗試題零分為止。

Notation: In the following problems, $\mathbb{R}$ is the usual set of all real numbers. We will use underlined letters such as $\underline{a} \in \mathbb{R}^n$ to denote a real, column vector $\underline{a}$ of length n and similarly will use boldface letters such as $\mathbf{A} \in \mathbb{R}^{m \times n}$ to denote a real matrix $\mathbf{A}$ of size $m \times n$. $\underline{0}$ is the all-zero column vector of proper length. $\mathbf{A}^\top$ is the transpose of matrix $\mathbf{A}$. rank($\mathbf{A}$) denotes the rank of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix. det($\mathbf{A}$) and tr($\mathbf{A}$) are respectively the determinant and trace of square matrix $\mathbf{A}$. Unless otherwise stated, all vector spaces and linear combinations are over field $\mathbb{R}$ and the orthogonality is with respect to the usual Euclidean inner product. Primes of functions of one variable denotes the derivatives with respective the variable, for instance, $y'(x) = \frac{d}{dx}y(x)$.

1.

Let $\mathbf{R}$ be the reduced row echelon form of a matrix $\mathbf{A}$, where $\mathbf{A}$ and $\mathbf{R}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & \times & \times & 3 \\ 1 & \times & \times & 1 \\ 0 & \times & \times & -1 \\ -1 & \times & \times & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \times & \times & \times & 1 \\ \times & \times & \times & 1 \\ \times & \times & 1 & 0 \\ \times & \times & \times & 0 \end{bmatrix},$$

and $\times$ represents an unknown number. Which of the following statements is/are true?

[A] It is possible that $\underline{a}_3$ is a zero vector.

[B] The rank of A can be less than 3.

[C] $\sum_{i=1}^{4} a_{i,2} = 2$, where $a_{i,j}$ is the (i,j)-th entry of $\mathbf{A}$.

[D] The unknown numbers in R can be uniquely determined.

[E] None of the above.

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2.

Continue from the previous problem. Which of the following statements is/are true?

[A] The column space of $\mathbf{A}$ is the same as the row space of $\mathbf{R}$.

[B] We can obtain a basis for null($\mathbf{A}$) from $\mathbf{R}$.

[C] Let $\mathbf{B} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_4 \end{bmatrix}$, where $\underline{a}_i$ is the i-th column vector of $\mathbf{A}$. Then the reduced row echelon form of $\mathbf{B}$ can be uniquely determined.

[D] The rank of the matrix $\begin{bmatrix} \mathbf{A} \\ \mathbf{R} \end{bmatrix}$ is the same as the rank of $\mathbf{A}$.

[E] None of the above.

3.

Let $\mathbf{A}$ be $m \times n$ and $\underline{b}$ be $m \times 1$. Consider $\mathbf{A}\underline{x} = \underline{b}$. Which of the following statements is/are true?

[A] If rank $(\mathbf{A}) < n$, there are infinitely many solutions.

[B] It is always true row($\mathbf{A}^T\mathbf{A}$)=row($\mathbf{A}$).

[C] The minimum of $||\mathbf{A}\underline{x} - \underline{b}||$ is the distance between $\underline{b}$ and null($\mathbf{A}$).

[D] The solution that minimizes $||\mathbf{A}\underline{x} - \underline{b}||$ is unique.

[E] None of the above.

4.

Let $\mathbf{A}$ be an $n \times n$ matrix. Which of the following statements is/are true?

[A] The eigen values of $\mathbf{A}\mathbf{A}^T$ are identical to those of $\mathbf{A}^T\mathbf{A}$.

[B] $\mathbf{A}\mathbf{A}^T$ can always be diagonalized.

[C] If $\mathbf{A}$ is similar to a diagonal matrix, then we can use the eigen vectors of $\mathbf{A}$ to form a basis for $\mathbb{R}^n$.

[D] rank (null ($\mathbf{A}$))=k if and only $\mathbf{A}$ has $n-k$ nonzero eigen values.

[E] None of the above.

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5.

Let V be the vector space of the zero polynomial and all polynomials of degree less than or equal to 2, with the operations of polynomial addition and multiplication of a polynomial by a scalar. Let T be a linear transformation from V to V defined as

$$T(p(x)) = -p(0) + (p'(1) + p'(-1))x + p(1)x^2.$$

Which of the following statements is/are true?

[A] T is not an isomorphism.

[B] The dimension of the range space of T is 2.

[C] The eigen values of T are 1 and 2.

[D] We can use the eigen vectors of T to form a basis for $\mathbb{R}^3$.

[E] None of the above.

6. Given that $\mathbf{W} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(\mathbf{W}) = 5$, which of the following answers is/are true?

[A] $\det(-3\mathbf{W}) = -15$

[B] $\det(\mathbf{W}^{-1}) = -5$

[C] $\det(\mathbf{W}^2) = 25$

[D] $\det((4\mathbf{W}^{-1})^T) = 4/5$

[E] None of the above.

7.

Let $V = R^{2\times 2}$, $\mathbf{W}_1 = \left\{\begin{bmatrix} p & q \\ r & p \end{bmatrix} \in V \middle| p, q, r \in R\right\}$ and $\mathbf{W}_2 = \left\{\begin{bmatrix} 0 & p \\ -p & q \end{bmatrix} \in V \middle| p, q \in R\right\}$. Which of the following statements is/are true?

[A] $\mathbf{W}_1$ is a subspace of V.

[B] $\mathbf{W}_2$ is a subspace of V.

[C] Dimension of the sum of the two subspaces $\mathbf{W}_1 + \mathbf{W}_2$ is 3.

[D] Dimension of $\mathbf{W}_1 \cap \mathbf{W}_2$ is 1.

[E] None of the above.

8.

Which of the following statements is/are true?

[A] If V is the vector space of all polynomials over R, then V is infinite-dimensional.

[B] If V and W are subspaces of R^{13}, $\dim(V) = 7, \dim(W) = 8$, then the smallest possible dimension of $V \cap W$ is 1.

[C] If V and W are subspaces of R^{13}, $\dim(V) = 7, \dim(W) = 8$, then the smallest possible dimension of $V + W$ is 7.

[D] If $\mathbf{P} \neq \mathbf{0}$ is a skew-symmetric matrix and $\mathbf{Q} \neq \mathbf{0}$ is a symmetric matrix in the vector space of $n \times n$ matrices, then $\mathbf{P}$ and $\mathbf{Q}$ are linearly independent.

[E] None of the above.

9.

Suppose that $T: R_2[x] \to R_3[x]; T\big(f(x)\big) = xf(x) + f'(x)$, where $R_n[\,]$ denotes the vector space of polynomials with real number coefficients, and n denotes the largest degree. Which of the following statements is/are true?

[A] Dimension of range of T is 4.

[B] T is a linear transformation.

[C] T is one-to-one.

[D] T is onto.

[E] None of the above.

10. $\mathbf{A} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times n}$, $\mathbf{C} \in R^{n\times n}$, which of the following is/are true?

[A] If $\mathbf{A}$ is singular and $\mathbf{B}$ is invertible, then $\mathbf{A} + \mathbf{B}$ is invertible.

[B] $\det(\mathbf{ABC}) = \det(\mathbf{BAC})$

[C] If the entries of $\mathbf{A}$ and its inverse $\mathbf{A}^{-1}$ are all integers, then $\det(\mathbf{A}) = 1$.

[D] $rank(\mathbf{AB}) = rank(\mathbf{BA})$

[E] None of the above.

11. The differential equation $y'' - 5y' + 6y = 5sin(t) - 5cos(t)$ has the following solution:

$y = f(t) + a \cdot e^{bt} + c \cdot e^{3t}$

where:

$f(t)$ is a function of t

a and c are some arbitrary coefficients

b is an integer

y is a function of t. Which of the following can be correct?

[A] $f(t) = cos(t)$

[B] $f(t) = sin(t)$

[C] $b = 1$

[D] $b = 2$

[E] None of the above.

12. Given the following differential equation:

$mu''(t) + \gamma u'(t) + ku(t) = 0$

and assuming the characteristic equations give:

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Assuming A and B are arbitrary coefficients. Which of the following are correct?

[A] If $\gamma^2 - 4mk > 0$, then $u(t) = Ae^{r_1 t} + Be^{r_2 t}$, where $r_1 < 0$ and $r_2 < 0$

[B] If $\gamma^2 - 4mk = 0$, then $u(t) = (A + Bt)e^{-\gamma t/2m}$, where $\gamma/2m > 0$

[C] If $\gamma^2 - 4mk < 0$, then $u(t) = [Acos(\mu t) + Bsin(\mu t)]e^{-\gamma t/2m}$, where $\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} > 0$

[D] In all cases, $\lim_{t \to \infty} u(t) = 0$

[E] None of the above.

13. When we solve the equation $df(x,y) = (2x-y)dx + (2y-x)dy = 0$, we know the following is true:

$$\frac{\partial(2x-y)}{\partial y} = -1$$

$$\frac{\partial(2y-x)}{\partial x} = -1$$

Therefore, the differential equation is exact. Solving the differential equation gives the solution:

$f(x,y) = ax^2 - bxy + cy^2 + d =$ constant

where $a = 1, b$ and c are integers, d is some arbitrary constant. Which of the following is true?

[A] $a+b+c=1$

[B] $a+b+c=2$

[C] $a+b+c=3$

[D] $a+b+c=4$

[E] None of the above.

14. Find a series solution of Airy's equation about $x = 0$:

$y'' - xy = 0$

y is a function of x, so y can also be written as $y(x)$. We know $y(0) = 1, y'(0) = 0$. The series solution can be assumed to have the following form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

which of the following is a solution for $y(x)$?

[A] $y(x) = 1 + \sum_{n=1}^{\infty} \dfrac{x^{3n+1}}{(3\cdot4)\cdot(6\cdot7)\ldots(3n)\cdot(3n+1)}$

[B] $y(x) = x + \sum_{n=1}^{\infty} \dfrac{x^{3n}}{(2\cdot3)\cdot(5\cdot6)\cdot\ldots(3n-1)\cdot(3n)}$

[C] $y(x) = x + \sum_{n=1}^{\infty} \dfrac{x^{3n+1}}{(3\cdot4)\cdot(6\cdot7)\ldots(3n)\cdot(3n+1)}$

[D] $y(x) = 1 + \sum_{n=1}^{\infty} \dfrac{x^{3n}}{(2\cdot3)\cdot(5\cdot6)\cdot\ldots(3n-1)\cdot(3n)}$

[E] None of the above.

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15.

Solve the differential equation:

$y''(t) + 4y(t) = g(t)$, with the condition $y(0) = 0$ and $y'(0) = 0$

where

$g(t) = 0$, if $0 \le t < 5$

$g(t) = \frac{t-5}{5}$, if $5 \le t < 10$

$g(t) = 1$, if $10 \le t$

The solution is $y(t) = \frac{1}{5}u_5(t) \cdot f(t-5) - \frac{1}{5}u_{10}(t) \cdot f(t-10)$

where $u_a(t)$ is function defined as $u_a(t) = 0$ when $t < a$ and $u_a(t) = 1$ when $t \ge a$

What is the function $f(t)$?

[A] $f(t) = \frac{1}{2}t - \frac{1}{4}sin(2t)$

[B] $f(t) = \frac{1}{4}t - \frac{1}{8}sin(2t)$

[C] $f(t) = \frac{1}{8}t - \frac{1}{16}sin(2t)$

[D] $f(t) = \frac{1}{16}t - \frac{1}{32}sin(2t)$

[E] None of the above.

16.

Given the differential equation $y'(t) + 3t^2y(t) = 5t^2$, the solution is:

$$y(t) = a - \frac{b}{e^{f(t)}}$$

Where a is a real number, b is an arbitrary coefficient, and $f(t)$ is a function of t. Which of the following can be correct?

[A] $a = \frac{5}{3}$

[B] $a = \frac{5}{4}$

[C] $f(t) = t^2$

[D] $f(t) = t^3$

[E] None of the above.

17.

We try to classify differential equations based on their order, linear vs. linear, and whether they're ordinary differential equations (ODE) or partial differential equations (PDE).

Differential Equation	Order	Linear vs. Nonlinear	ODE vs. PDE
$p''(t) = -mg$	a	b	c
$y'(t) = y(t)(1 - y(t))$	d	e	f
$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$	g	h	i
$\frac{\cos(t)}{1+t^2}y'''''' - 2y' + (1+t)^{10}y = 0$	j	k	l
$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 1 + \frac{\partial^2 u}{\partial x^2}$	m	n	o

Please assign values to the variables $(a - o)$ based on the following rules:

Order: the number corresponds to the order of the differential equation

Linear vs. Nonlinear: assign 1 for linear, and 0 for nonlinear

ODE vs. PDE: assign 1 for ODE, and 0 for PDE

Take this equation for example:

$$p''(t) = -mg$$

It is a second order linear ODE. Hence, $a = 2, b = 1, c = 1.$

Which of the following are correct?

[A] $d + e + f = 2$

[B] $g + h + i = 3$

[C] $j + k + l = 7$

[D] $m + n + o = 4$

[E] None of the above.

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18.

In this problem, we treat y as a function of x. Assign the variables (a, b, c, d) as 1 for "True" and 0 for "False" for each of the following description regarding ordinary point and singular point.

(a) $x = 0$ is an ordinary point for $x^2y'' + 2y' + sin(x) \cdot y = 0$

(b) $x = 2$ is a singular point for $(x - 2)y'' + x^2y' + 5y = 0$

(c) $x = 1$ is a singular point for $(x^2 - 1)y'' + (1 - x)y' + (x^2 - 2x + 1)y = 0$

(d) $x = 0$ is an ordinary point for $y'' - xy' + 4y = 0$

[A] $a + b + c + d = 1$

[B] $a + b + c + d = 2$

[C] $a + b + c + d = 3$

[D] $a + b + c + d = 4$

[E] None of the above.

19. Solve the differential equation:

$y''(t) + y(t) = sin(2t)$, with the condition $y(0) = 2$ and $y'(0) = 1$

The solution is $y(t) = a \cdot f(t) + b \cdot g(t) - \frac{1}{3}sin(2t)$

where $f(t)$ is an even function, and $g(t)$ is an odd function. Both $f(t)$ and $g(t)$ don't have any real number coefficients. a and b are real numbers. Which of the following can be true?

[A] $a = 1$

[B] $a = 2$

[C] $b = \frac{5}{3}$

[D] $b = \frac{1}{3}$

[E] None of the above.

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20.

For the following pair of differential equations:

$$\frac{dx}{dt} = 1.2 + \frac{y}{100} - \frac{7x}{100}$$

$$\frac{dy}{dt} = \frac{3x}{100} - \frac{3y}{100}$$

With initial conditions $x(0) = 0$ and $y(0) = 20$. The solutions are:

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t} + C_3$$

$$y = D_1 e^{r_1 t} + D_2 e^{r_2 t} + D_3$$

where $C_1, C_2, C_3, D_1, D_2, D_3$ are real numbers. Which of the following can be correct?

[A] $C_1 = -5(2 - \sqrt{7})$

[B] $C_2 = -5(2 + \sqrt{7})$

[C] $D_1 = -5(2 - \sqrt{7})$

[D] $D_2 = -5(2 + \sqrt{7})$

[E] None of the above.