國立中央大學114學年度碩士班考試入學試題

系所： 數學系 碩士班 應用數學組(一般生)
數學系 碩士班 應用數學組(在職生)

科目： 微積分

＊本科考試禁用計算器

甲 (填充題) ：共 8 題，每題 9 分。請將答案依題號順序寫在答案卷上，不必寫演算過程。

1. (9%) $\lim_{x\to\infty}(2x)^{\frac{3}{x}} =$ __________.

2. (9%) $\displaystyle\sum_{n=2}^{\infty}\frac{n+2}{n(n+1)2^{n+1}} =$ __________.

3. (9%) Let $f(x) = (x^2+1)^2\cdot\operatorname{sech}(\ln x)$. Then $f'(1) =$ __________.

4. (9%) $\displaystyle\int_0^{\arctan 3}\frac{1}{1+2\cos^2\theta}\,d\theta =$ __________.

5. (9%) The region bounded by the curves $y = x^2$ and $x = y^2$ is revolved about the line $y = 1$ to generate a solid. Then the volume of the solid is equal to __________.

6. (9%) The maximum value of $f(x,y) = x^2+y^2+4x-6y$ on the domain defined by $x^2+y^2\le 16$ is equal to __________.

7. (9%) $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3}\frac{e^{2x}\sin(\pi y^2)}{y^2}\,dx\,dy\,dz =$ __________.

8. (9%) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ oriented counter-clockwise. Then $\displaystyle\oint_C xy\,dx + x^2y^3\,dy =$ __________.

乙 (計算、證明題) ：共 2 大題，共 28 分。須詳細寫出計算及證明過程，否則不予計分。

1. Let P be the intersection curve of the surface $z^2 = x^2+y^2$ and the plane $x = 2z+3$.
   (i) (6%) Find the minimum distance from the origin $(0,0,0)$ to the curve P.
   (ii) (6%) Find the maximum distance from the origin $(0,0,0)$ to the curve P.

2. Prove Newton's binomial theorem by the following steps: Let $\alpha\in\mathbb{R}$ that is not a non-negative integer. Newton's binomial theorem states that for any $x\in(-1,1)$ the following equation holds:

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n, \tag{1}$$

where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{n} = \displaystyle\prod_{k=1}^{n}\frac{\alpha-k+1}{k}$ for any integer $n\ge 1$. Let $f(x)$ be the right-hand side of (1).

   (i) (3%) Prove that the radius of convergence of the power series $f(x)$ is 1.
   (ii) (5%) Prove that $(1+x)f'(x) = \alpha\cdot f(x)$ for all $x\in(-1,1)$.
   (iii) (8%) Prove that the equation (1) holds for all $x\in(-1,1)$.