參考用

Q1 [18%] Let A be a 3×3 matrix with $A^3 = A^2$.
  (a) [4%] Find all possible eigenvalues of A.
  (b) [8%] Find all possible characteristic polynomials of A.
  (c) [6%] Find all possible Jordan forms of A.

Q2 [16%] Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ -1 & -1 & 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.
  (a) [8%] Show that $A\vec{x} = \vec{v}$ has no solution.
  (b) [8%] Find the least square solution of $A\vec{x} = \vec{v}$. (Hint: Find $\vec{x}$ such that $\|A\vec{x} - \vec{v}\|^2 = (A\vec{x} - \vec{v})^t(A\vec{x} - \vec{v})$ is minimal, where the superscript t means vector transpose.)

Q3 [16%] Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.
  (a) [8%] Write $A = PDP^{-1}$, where D is a diagonal matrix.
  (b) [8%] Evaluate

$$e^A = \sum_{n=1}^{\infty} \frac{1}{n} A^n.$$

Q4 [10%] Evaluate

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{4 - x^2 - y^2}\, dx\, dy.$$

Q5 [15%] Find the minimum and maximum of the function $x^2 + 2y^2$
  (a) [10%] on the circle $x^2 + y^2 = 4$.
  (b) [5%] in the disc $x^2 + y^2 \leq 4$.

Q6 [15%] Let $\{a_n\}$ be a sequence of nonnegative numbers.
  (a) [10%] Show that if $\sum a_n$ converges, then $\sum a_n^2$ also converges.
  (b) [5%] If $\sum a_n^2$ converges, does $\sum a_n$ also converge? Prove or disprove your answer.

Q7 [10%] Let $f(x)$ be a continuous function on $[0,1]$. Suppose $f(0) \leq 0$ and $f(1) \geq 1$. Prove that for every positive integer n, there exists $a_n \in [0,1]$ such that $f(a_n) = (a_n)^n$.