參考用

科目：應用數學(2001)

校系所組：中央大學光電科學與工程學系照明與顯示科技碩士班
交通大學電子物理學系（丙組）
交通大學物理研究所
清華大學物理學系
清華大學先進光源科技學位學程（物理組）
清華大學材料科學工程學系（乙組）
陽明大學生醫光電研究所（理工組）
中央大學天文研究所
清華大學天文研究所

1. (20%) A vector field is given by: $\mathbf{A} = (2x-y)\,\mathbf{i} - y^2z^3\,\mathbf{j} - y^3z^2\,\mathbf{k}$. Verify the Stoke's theorem, given that S is the upper half surface of the sphere of radius one centered at the origin and C is its boundary.

2. Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) (6%) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}$.
(b) (3%) Find an orthogonal matrix $\mathbf{X}$ that diagonalizes $\mathbf{A}$ to a diagonal matrix $\mathbf{D}$.
(c) (3%) Express $\mathbf{A}^{-1}$ in terms of $\mathbf{X}$ and $\mathbf{D}$.
(d) (3%) Find the determinant of $\mathbf{A}^4$.

3. Consider a linear-regression problem $y = ax + b$, where the slope a and the intercept b are parameters to be estimated. Measurements were performed at x = -1, 0, 1, 2 and the resulting y coordinates are y = 0.7, 1.6, 2.2, 3.0 respectively. Hence this problem can be modeled as a linear system: $\mathbf{A}\begin{bmatrix} a \\ b \end{bmatrix} = \bar{y} = \begin{bmatrix} 0.7 \\ 1.6 \\ 2.2 \\ 3.0 \end{bmatrix}$.

(a) (3%) Construct the system matrix $\mathbf{A}$.
(b) (2%) How would you describe this system? (multiple choices)
(A) Underdetermined; (B) Determined; (C) Overdetermined;
(D) Consistent; (E) Inconsistent.
(c) (5%) The pseudoinverse of $\mathbf{A}$ is $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$, where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. Compute $\mathbf{A}^T\mathbf{A}$ and $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$. Find the least-squares solution for $\begin{bmatrix} a \\ b \end{bmatrix}$ as $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\bar{y}$.

(d) (5%) Another way to find the least-squares solution for $\begin{bmatrix} a \\ b \end{bmatrix}$ is to define a cost function $Q = \sum_{i=1}^{4}\left[(ax_i + b) - y_i\right]^2$ and find $\begin{bmatrix} a \\ b \end{bmatrix}$ by setting $\frac{\partial Q}{\partial a} = 0$ and $\frac{\partial Q}{\partial b} = 0$. Determine $\begin{bmatrix} a \\ b \end{bmatrix}$ this way.

注意：背面有試題

科目：應用數學(2001)　　校系所組：中央大學光電科學與工程學系照明與顯示科技碩士班
交通大學電子物理學系（丙組）
交通大學物理研究所
清華大學物理學系
清華大學先進光源科技學位學程（物理組）
清華大學材料科學工程學系（乙組）
陽明大學生醫光電研究所（理工組）
中央大學天文研究所
清華大學天文研究所

4. (15%) A string is clamped at both ends $x=0$ and $x=L$. We assume that the vibration is of small amplitude and satisfies the wave equation,

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} y(x,t),$$

where v is the wave velocity. The string is set in vibration with the following initial conditions:

$$y(x,0)=0,$$

$$\frac{\partial y(x,t)}{\partial t} = Lv_0\delta(x-a) \quad \text{at} \quad t=0,$$

where $v_0 = constant$. Solve for $y(x,t)$. [Hint: let $y(x,t) = X(x)\,T(t)$]

5. The Laplace transform of $N(t)$ is denoted as

$$\mathcal{L}[N(t)] = \int_0^\infty e^{-st} N(t)\,dt = F(s).$$

(a) (5%) Show that $\mathcal{L}\left[\frac{dN(t)}{dt}\right] = sF(s) - N(0)$.

(b) (5%) Show that $\mathcal{L}^{-1}\left(\frac{1}{s+\lambda}\right) = e^{-\lambda t}$.

(c) (10%) Three radioactive nuclei decay successively in series such that the number $N_i(t)$ of three types obey the equations:

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1$$

$$\frac{dN_2(t)}{dt} = -\lambda_1 N_1 - \lambda_2 N_2$$

$$\frac{dN_3(t)}{dt} = -\lambda_2 N_2 - \lambda_3 N_3$$

If initially $N_1 = N$, $N_2 = 0$, and $N_3 = n$, find $N_3(t)$ by using the Laplace transform.

科目：應用數學(2001)　　校系所組：中央大學光電科學與工程學系照明與顯示科技碩士班
交通大學電子物理學系（丙組）
交通大學物理研究所
清華大學物理學系
清華大學先進光源科技學位學程（物理組）
清華大學材料科學工程學系（乙組）
陽明大學生醫光電研究所（理工組）
中央大學天文研究所
清華大學天文研究所

參考用

6. Denote the Fourier transform of $y(x)$ as $\mathcal{F}[y(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x)\, e^{i\lambda x} dx = F(\lambda)$. Given

$$\frac{d^2 y(x)}{dx^2} - y(x) = -\theta(1-|x|),$$

where $-\infty < x < \infty$ with $y(x) \to 0$ and $\frac{dy}{dx} \to 0$ as $|x| \to 0$.

Note $\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ is the Heaviside step function.

(a) (5%) Apply Fourier transform to $\frac{d^2 y(x)}{dx^2} - y(x) = -\theta(1-|x|)$.

(b) (10%) Use the method of Fourier transform to solve the above differential equation.