國立中央大學111學年度碩士班考試入學試題

所別： 資工類

科目： 離散數學與線性代數

離散數學

第1～10多選題，ABCDE每一選項1分，答錯一個選項倒扣1分；答錯倒扣到第1~10題零分為止。（每題5分）

1. Let $Q(x, y)$ be the statement "$x + y = x - y$," which of the following are true?
   (A) $\exists x \exists y Q(x, y)$
   (B) $\forall x \exists y Q(x, y)$
   (C) $\exists y \forall x Q(x, y)$
   (D) $\forall y \exists x Q(x, y)$
   (E) $\forall x \forall y Q(x, y)$

2. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 p.m., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. There is a period of consecutive hours during which
   (A) the arm wrestler had exactly 2 matches.
   (B) the arm wrestler had exactly 23 matches.
   (C) the arm wrestler had exactly 24 matches.
   (D) the arm wrestler had exactly 25 matches.
   (E) All the above statements are incorrect.

3. In the RSA public key system,
   (A) its correctness is based on Fermat's theorem and Chinese remainder theorem.
   (B) two large primes are chosen so that one acts as the public key and the other the private key.
   (C) $(M^e)^d = (M^d)^e = M$, where M represents a message, e represents the public key, and d the corresponding private key.
   (D) if Alice wants to send a message to Bob and prove her identity, Alice first generates a hash value from the message and encrypts the hash value by her own public key and then sends the plaintext message and the encrypted hash value to Bob. After Bob receives the message, he decrypts the hash value by Alice's private key. Besides, he also generates a hash value from the plaintext message. If both values match, it proves the message comes from Alice.
   (E) all of the above statements are incorrect.

4. Let G be a connected planar simple graph with e edges and v vertices, then
   (A) $r = e + v - 2$, where r represents the number of regions in a planar representation of G.
   (B) $e \leq 3v - 6$ when $v \geq 3$
   (C) G has a vertex of degree not exceeding five
   (D) A graph is nonplanar if it contains a subgraph homeomorphic to $K_{2,3}$
   (E) A graph is nonplanar if it contains a subgraph homeomorphic to K_5

5. A full m-ary tree with
   (A) n vertices has $i = (n - 1)/m$ internal vertices
   (B) n vertices has $l = [(m - 1)n + 1]/m$ leaves
   (C) i internal vertices has $l = (m - 1)i + 1$ leaves
   (D) l leaves has $n = (ml - 1)/(m - 1)$ vertices
   (E) l leaves has $i = (l - 1)/(m - 1)$ internal vertices.

注意:背面有試題

6. Suppose $|A| = 8, |B| = 3$. Consider the number of possible functions $f : A \to B$
   (A) The number of possible f is 3^8.
   (B) If f is one-to-one, the number is 6.
   (C) If f is onto , the number is 8^3.
   (D) The number of possible f^{-1} is 0.
   (E) If f is bijection, the number is 0.

7. Consider a predicate $P(i,j)$, where i,j are elements from the same domain S. Suppose all following statements are true:
   $\forall i, p(i,i)$; $\forall i,j,(i \neq j), p(i,j) \to \neg p(j,i)$; $\forall i,j,k, \big(p(i,j) \wedge p(j,k)\big) \to p(i,k)$;
   $\forall i,j, p(i,j) \vee p(j,i)$; $\exists! k, \forall i \neq k, p(k,i)$;
   ($\exists!$ means "exist one and only one")
   Which of the following options are correct?
   (A) p describes an equivalence relation.
   (B) p describes a partial order set.
   (C) p describes a total order relation.
   (D) p describes a lattice .
   (E) p describes a well order relation.

8. Consider a general recursive mechanism ***R***, whose input is an array **A**() initially with n elements. The pseudo code of R is listed below, It takes a parameter k, use $\sqrt{n}$ (n is the size of A when applied, n may change) steps to partition **A**, and may call another function Q, whose complexity may vary and depends on the size of its input.

   Procedure *R* (array A with size n, int k)
   1. if $n<k$ exit;
   2.partition A into k equal size parts $A_1, A_2, \dots, A_k$, (This takes $\sqrt{n}$ steps)
   3. for (i=1 to k)
   . call ***R***(A_k,k); // may change A_k;
   4. merge $A_1, A_2, \dots, A_k$ into new A with size n; (This takes k steps)
   5. call Q(A);
   6. return()

   There are 3 derived algorithms X,Y,Z, based on R. X sets k as 2, and its Q's complexity is $\theta(m)$, m is the size of Q's input. Y sets k as 4, and its Q's complexity is $\theta(\sqrt{m})$, m is input size. Z sets k as 9, and its Q's complexity is $\theta(m)$, m is input size.
   What following comparisons are true?
   (A) X's complexity is better than Y and Z.
   (B) Y's complexity is better than X and Z.
   (C) X and Y's complexities are the same.
   (D) X and Z's complexities are the same.
   (E) X's complexity is $\theta(n \log n)$.

9. Concerning the binomial coefficient $\binom{r}{k}$, what following formula are true?
   (A) $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}\ integer\ k \neq 0$;
   (B) $\sum_0^n \binom{n}{k}(-1)^k = 0, integer\ n \geq 0$;
   (C) $\sum_0^n \binom{k}{m} = \binom{n+1}{m+1}, integer\ m,n \geq 0$;
   (D) $\sum_0^n \binom{r+k}{k} = \binom{r+n+1}{n}, integer\ n;$.
   (E) $\binom{r}{k} = \binom{r-1}{k-1} + \binom{r}{k-1}, integer\ k$;

注意:背面有試題

10. There is a series of number a_n ,. $\forall n > 1, (a_{n-1} - a_{n-2}) = 2(a_n - a_{n-1})$, a_0=5, a_1=9. Using generating function *f(z)* to solve a_n, what of the followings are true?

(A) $f(z) = (\frac{10+3z}{2-3z+z^2})$

(B) $f(z) = (\frac{1}{1-2z}) + (\frac{z}{(1-z)^2(1-2z)})$.

(C) $f(z) = (\frac{-8}{1-(1/2)z}) - (\frac{13}{(1-z)})$.

(D) $a_n = (13 - 8 \times (\frac{1}{2})^n)$

(E) $a_n = 16 \times 2^n - 13$.

線性代數

第11～12, 16～20題為多選題，ABCDE每一選項1分，答錯一個選項倒扣1分，答錯倒扣到第11～12, 16～20題零分為止。（每題5分）

11. Which of the following statements are true?
(A) If $\boldsymbol{A}^2$ is a diagonal matrix, $\boldsymbol{A}$ must be a diagonal matrix.
(B) If $\boldsymbol{A}_{n\times n}$ has a complete set of orthonormal eigenvectors (*i.e.*, *n* eigenvectors that are orthonormal), it must be a Hermitian matrix.
(C) If $\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{A}^H\boldsymbol{A}$, where $\boldsymbol{A}^H$ is the Hermitian transpose of $\boldsymbol{A}$, then $\boldsymbol{A}$ can always be diagonalized.
(D) If the rank of an *n* by *n* matrix, $\boldsymbol{A}$, is $r < n$, then its *n-r* eigenvalues are zero.
(E) If *k* is an eigenvalue of $\boldsymbol{A}$, we may find more than one eigenvector corresponding to *k*.

12. For the 197 by 197 finite difference matrix, $\boldsymbol{A}$, which statements are true?

$$\boldsymbol{A}_{197\times197} = \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & 0 & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{bmatrix}$$

(A) The determinant of $\boldsymbol{A}$ is *D*. Then, mod(|*D*|,5)=3, where mod(.) is the modulo operation.
(B) $\boldsymbol{A}$ can be expressed as $\boldsymbol{A}=\boldsymbol{B}+\boldsymbol{C}$, where $\boldsymbol{B}$ is a symmetric matrix and $\boldsymbol{C}$ is a skew-symmetric matrix.
(C) $\boldsymbol{A}$ can be expressed as $\boldsymbol{A}=\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}}$, where $\boldsymbol{B}$ is an orthogonal matrix and $\boldsymbol{C}$ is a diagonal matrix.
(D) $\boldsymbol{A}$ can be expressed as $\boldsymbol{A}=\boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}}$, where $\boldsymbol{B}$ is a lower triangular matrix and $\boldsymbol{C}$ is a diagonal matrix.
(E) $\boldsymbol{A}$'s smallest eigenvalue is negative.

注意:背面有試題

第 13～15 單選題，每題 5 分答錯倒扣 2 分，扣到所有單選題零分為止。

13. $\boldsymbol{A}^5=\begin{bmatrix}32 & 100000\\ 0 & 3125\end{bmatrix}$. The determinant of $\boldsymbol{A}$ is M. The closest integer of summing up the four entries in $\boldsymbol{A}$ is $N = \text{round}(\boldsymbol{A_{11}} + \boldsymbol{A_{12}} + \boldsymbol{A_{21}} + \boldsymbol{A_{22}})$. What is the value of $\text{mod}(M + N \times 13, 5)$, where mod(.) is the modulo operation.
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

14. First, consider the linear system

$$\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$$

If k = a, the system has more than one solution. If k = b, the system has no solution. The closest integer of |a-b| is c. M=mod(c,5), where mod(.) is the modulo operation.
Next, consider the linear system

$$\begin{cases} x + y + z = 4 \\ 2x + y + 2z = 7 \\ x - 2z = 3 \\ y + 2z = 4 \end{cases}$$

Its least squares solution is $\{\tilde{x}, \tilde{y}, \tilde{z}\}$. The closest integer of $(\tilde{y} \times 50)$ is N.
What is the value of mod(M+N,5),?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

15. The determinant of the following matrix $\boldsymbol{A}$ is M.

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 8 & 4 & 2 \\ 2 & 6 & 0 & -4 & 3 \\ 2 & 0 & 2 & 6 & 2 \\ 0 & 2 & 8 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \end{bmatrix}$$

The determinant of the following matrix $\boldsymbol{B}$ is N.

$$\boldsymbol{B} = \begin{bmatrix} 1.25 & 1 & 1 & 1 \\ 1 & 1.2 & 1 & 1 \\ 1 & 1 & 1.1 & 1 \\ 1 & 1 & 1 & 1.5 \end{bmatrix}$$

$K = \left|\frac{M}{N}\right| \times \frac{1}{10000}$. Which of the following is the range of K?
(A) $0 \le K < 1$ (B) $1 \le K < 2$ (C) $2 \le K < 3$ (D) $4 \le K < 5$ (E) $5 \le K$

16. $\boldsymbol{A}$ and $\boldsymbol{B}$ are two square matrices.
(A) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then det ($\boldsymbol{A}$) = det ($\boldsymbol{B}$).
(B) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same eigenvectors.
(C) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$.
(D) If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $\boldsymbol{AB}$ is similar to $\boldsymbol{BA}$.
(E) If $\boldsymbol{A}^2$ is similar to $\boldsymbol{B}^2$, then $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.

注意:背面有試題

17. For a $n \times n$ real square matrix $\boldsymbol{A}$. λ_i's are eigenvalues of $\boldsymbol{A}$.
    (A) λ_i's may be complex number.
    (B) If some $\lambda_i = 0$, then $\boldsymbol{A}$ is always not invertible.
    (C) If some λ_i's are the same, then $\boldsymbol{A}$ is not diagonalizable.
    (D) If $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$, then λ_i's are positive real.
    (E) If $\boldsymbol{A}$ is diagonalizable, then $\boldsymbol{A}$ has many different matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$.

18. If $\boldsymbol{A}$ is a $m \times n$ matrix and $\boldsymbol{A}$ can be $\boldsymbol{QR}$ factorized. $\boldsymbol{W}$ is the set of all columns of $\boldsymbol{A}$, and $\boldsymbol{W}^{\perp}$ is the orthogonal complement of $\boldsymbol{W}$, then
    (A) $\boldsymbol{W}$ is a subspace.
    (B) $\boldsymbol{W}^{\perp}$ is a subspace.
    (C) $(\boldsymbol{W}^{\perp})^{\perp} = \boldsymbol{W}$.
    (D) $\boldsymbol{W} \cap \boldsymbol{W}^{\perp} = \phi$ (empty set)
    (E) $m \geq n$.

19. In the following terminologies which are **not** related to the least-squares problem.
    (A) Inconsistent system.
    (B) Eigenvector.
    (C) Diagonalizable.
    (D) Orthogonal projection.
    (E) Gram - Schmidt process

20. Construct a spectral decomposition of matrix $\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. Which values are **not** in the solution matrices.
    (A) $\frac{1}{5}, \frac{-1}{5}$. (B) $\frac{2}{5}, \frac{-2}{5}$. (C) $\frac{4}{5}, \frac{-4}{5}$. (D) $\frac{1}{5}, \frac{4}{5}$. (E) $\frac{-1}{5}, \frac{-4}{5}$.