國立中央大學114學年度碩士班考試入學試題

系所： 資工類　　

科目： 離散數學與線性代數

＊本科考試禁用計算器

第一部分: 共20分，單選題，每題五分，錯一題倒扣2分，扣到單選題[整大題]0分為止

1. Given a linear transformation from R^2 to R^3 (with respect to standard bases) $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x-2y \\ 3x-4y \\ -7x \end{bmatrix}$. Now let $B_1 = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and $B_2 = \{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\}$ be non-standard bases for R^2 and R^3, respectively. If the matrix representation of the same linear transformation L (with respect to B_1 and B_2) is $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}_{B_1:B_2}$, what is $\lfloor |a+b+c+d+e+f| \rfloor \%5$? (% is the modulo operation. $\lfloor z \rfloor$ rounds z to the smaller nearest integer.)
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

2. Let the Gram-Schmidt QR factorization of the matrix $\begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$ be $\begin{bmatrix} \frac{1}{5} & \frac{-2}{5} & \frac{-4}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{-4}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{2}{5} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$. What is $\lfloor |a+b+c+d+e+f| \rfloor \%5$? (% is the modulo operation. $\lfloor z \rfloor$ rounds z to the smaller nearest integer.)
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

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3. Assume $p(x) = x^{10} - x^5 + 1$ and $A = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$. Let $p(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is $\lfloor |a + b + c + d| \rfloor \%5$? (% is the modulo operation. $\lfloor z \rfloor$ rounds z to the smaller nearest integer.)
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

4. Let $A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$ and the unitary matrix that diagonizes A be $U = \frac{1}{\sqrt{3}}\begin{bmatrix} a+ib & c+id \\ e+if & g+ih \end{bmatrix}$, what is the $\lfloor |a+c+e+g| \rfloor \%5$? (% is the modulo operation. $\lfloor z \rfloor$ rounds z to the smaller nearest integer.)
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) 4

第二部分: 共80分，多選題，每題5分，每選項單獨計分，答錯一個選項倒扣1分，扣到多選題[整大題]0分為止。

5. Which of the following vectors belong to the nullspace of $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$?
   (a) $(0,0,0,0)^T$
   (b) $(1,1,1,0)^T$
   (c) $(1,-2,1,0)^T$
   (d) $(-1,1,0,1)^T$
   (e) $(0,-1,1,1)^T$

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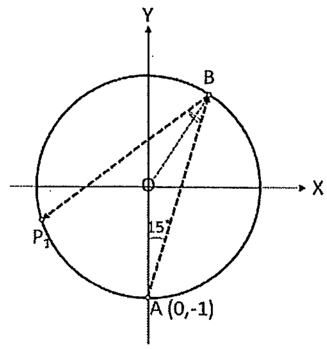
6. **A** and **B** are both N×N square matrices. Which of the following statements are true?
   (a) λ_A is the eigenvalue of **A** and λ_B is the eigenvalue of **B**. $\lambda_A \times \lambda_B$ is the eigenvalue of **AB**.
   (b) **AB** and **BA** have the same eigenvalues.
   (c) **AB** and **BA** have the same eigenvectors.
   (d) If $\mathbf{B}=\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, then **A** and **B** have the same eigenvalues.
   (e) If $\mathbf{A}\mathbf{A}^H=\mathbf{A}^H\mathbf{A}$ ($\mathbf{A}^H$ is **A**'s Hermitian Transpose), then **A** and $\mathbf{A}^H$ have the same eigenvalues.

7. **A** is an N×N symmetric matrix. Which of the following statements are true?
   (a) If $\mathbf{A}^{-1}$ exists, $\mathbf{A}^{-1}$ must be symmetric.
   (b) **A** must have a complete set of (N) orthonormal eigenvectors.
   (c) If $\mathbf{A}^2=\mathbf{A}$, its eigenvalues must be 0 and 1.
   (d) If the sum of each individual column of **A** is 1, then one of its eigenvalues is 1.
   (e) If **B** is also an N×N symmetric matrix. **AB** must be symmetric.

8. We are processing two-dimensional data on the X-Y plane. Which of the following are true?
   (a) One particle is initially located at the point A (0,-1) and on the unit circle. It goes straight to B on the unit circle as shown in the figure, and $\angle OAB=15^\circ$, where O is the origin (0,0). The particle is then reflected and reaches P_1 on the unit circle with $\angle OBA=\angle OBP_1$. The particle keeps reflecting and reaches point P_2 on the unit circle and so on.



   The coordinate of P_{120} is $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

   (b) The point A (0,-1) is linearly transformed by $\mathbf{J}=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$. After N linear transformations (i.e., $\mathbf{J}^N\begin{bmatrix}0\\ -1\end{bmatrix}$), the position is $Q_N=(x_N, y_N)$. $Q_{1001}=(x_{1001}, y_{1001})$, $x_{1001}=0$.
   (c) Following (B), the distance between Q_{1000} and Q_{1001} is M. The distance between Q_{999} and Q_{1000} is N. $\frac{M}{N} > 2.3$.

(d) $\mathbf{R}_K=\begin{bmatrix}\cos(K^\circ) & -\sin(K^\circ)\\ \sin(K^\circ) & \cos(K^\circ)\end{bmatrix}$, $\mathbf{M}_K=\begin{bmatrix}\cos(2\times K^\circ) & \sin(2\times K^\circ)\\ \sin(2\times K^\circ) & -\cos(2\times K^\circ)\end{bmatrix}$. A point C, (a,b), is on the unit circle and in the first Quadrant (0<a<1, 0<b<1). C is then linearly transformed by $\mathbf{Q}=\mathbf{R}_{15}\mathbf{M}_{15}\mathbf{R}_{30}\mathbf{M}_{30}\mathbf{R}_{45}$ (i.e, $\mathbf{Q}[a,b]^T$). The resultant point must be in the first Quadrant.

(e) $\mathbf{J}=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$, $\mathbf{T}=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$, $\mathbf{S}=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$. A unit circle U is transformed by **N=SJT** to become Z. (For a point D, (c,d), on the unit circle, we calculate $\mathbf{N}[c,d]^T$.) The sum of the largest and shortest distances between the origin O and Z is 25.

9. $\begin{bmatrix}a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\end{bmatrix}$ is processed by row operations to form $\begin{bmatrix}1 & 2 & 3 & 5\\ 2 & 5 & 7 & 13\\ 0 & 0 & 1 & 0\end{bmatrix}$. Which of the following are true?

(a) $\begin{vmatrix}d_1 & d_2 & d_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\begin{vmatrix}b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$

(b) After row operations, $\begin{bmatrix}a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\end{bmatrix}$ can become $\begin{bmatrix}2 & -1 & 0 & -5\\ 3 & 0 & -1 & -3\\ 0 & 3 & -2 & 9\end{bmatrix}$.

(c) If $\begin{cases}{a_1}^2+{a_2}^2+{a_3}^2=\ 2\\ {b_1}^2+{b_2}^2+{b_3}^2=\ 2\\ {d_1}^2+{d_2}^2+{d_3}^2=16\end{cases}$, then $|(a_1,a_2,a_3)\ \times\ (d_1,d_2,d_3)\,| = 4\sqrt{2}$, where $\times$ is the cross product.

(d) $\begin{cases}a_1x+c_1y+3b_1=d_1\\ a_2x+c_2y+3b_2=d_2\end{cases}$ has a unique solution.

(e) $\begin{cases}(3b_1+c_1)x-2a_1y+d_1z+2c_1=0\\ (3b_2+c_2)x-2a_2y+d_2z+2c_2=0\\ (3b_3+c_3)x-2a_3y+d_3z+2c_3=0\end{cases}$ has a unique solution.

10. Which of the determinants of the following matrices are zero?

(a) $\begin{bmatrix} 0 & 3 & 2 & 6 & 3 \\ -3 & 0 & 1 & 3 & 2 \\ -2 & -1 & 0 & 5 & 3 \\ -6 & -3 & -5 & 0 & 4 \\ -3 & -2 & -3 & -4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 2 & 4 & 3 \\ 5 & 4 & 1 & 3 & 2 \\ 2 & 5 & 2 & 5 & 3 \\ 2 & 6 & 4 & 8 & 6 \\ 4 & 7 & 8 & 7 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

For questions 11 and 12, matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ describes a binary relation **R.**

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11. About relation **R**, which of the following statements are true?

    (a) **R** is reflexive. (b) **R** is anti-symmetric (c) **R** is transitive

    (d) **R** is a partial ordering relation. (e) The transitive closure of **R** is symmetric.

12. Let **S** be the transitive closure of **R** and each element is a set.

    If **S** reflects operators' behavior on **Set**, what are possible operators?

    (a) $\cap$ (b) $\cup$ (c) $\subseteq$ (d) $-$ (set difference) (e) $\supset$

13. Consider the divisibility relation (i.e., |) and the poset R = (S, |), where S = {*1, 2, 3, 4, 6, 8, 12, 24*}. We can conclude that:

    (a) R is a total order.

    (b) R is a well order.

    (c) R has a greatest lower bound.

    (d) R is a lattice.

    (e) R has a maximal element.

14. In the following procedure P, P and B are both procedures; each statement line in and outside the loop counts 1 step.

Procedure P(array1[$a_1, a_2, \ldots, a_n$])

```
1. if n<9 exit.
2. call B(array1[a_1, a_2, ..., a_n])
   declare new empty array2, array3, array4;
3. for (i=1 to n)
4. { if ((i mod 9)=1)   insert a_i into array2;
5.   if ((i mod 9)=4)   insert a_i into array3;
6.   if ((i mod 9)=7)   insert a_i into array4 }
8. call P( array2 );
9. call P( array3 );
10. call P (array4);
```

In order to make P's complexity $O(n)$, which of the following can be B's complexity ?

(m is the input size of B)

(a) $O(m^{\frac{1}{2}})$ (b) $O(m^1)$ (c) $O(m^{\sqrt{2}})$ (d) $O(m^{\frac{1}{4}})$ (e) $O(m^2)$

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15. To solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}, a_0 = 0, a_1 = 1/2$, what of the followings are true? (generating function *f(z)* for this recurrence relation)

(a) $G(z)(1-4z+4z^2) = \frac{1}{1+z} - 1 + \frac{3}{2}z$ (b) $G(z)(1-4z+4z^2) = \frac{1}{1+z} - 1 + z$

(c) $G(z) = -\frac{1}{9}\frac{1}{1+z} + \frac{7}{9}\frac{z}{(1-2z)^2} + \frac{1}{9}\frac{1}{(1-2z)^2}$ (d) $G(z) = \frac{1}{9}\frac{1}{1+z} + \frac{19}{18}\frac{z}{(1-2z)^2} - \frac{1}{9}\frac{1}{(1-2z)^2}$

(e) $a_n = \frac{1}{9}(-1)^n + \frac{19}{36}n \cdot 2^n - \frac{1}{9}2^n$

16. Which of these graphs have an Euler circuit?

(a) C_5 (b) W_5 (c) K_5 (d) $K_{5,5}$ (e) Q_5

17. At a party, you and your partner meet three other couples. Several handshakes take place under the following rules: no one shakes hands with themselves or their own partner, no one shakes hands with the same person more than once, and the number of handshakes for the other seven people (excluding you) are all different. Based on these conditions, how many hands did you shake, and how many hands did your partner shake?

(a) You shook 4 hands.

(b) You shook 5 hands.

(c) You shook 3 hands.

(d) your partner shook 0 hand.

(e) your partner shook 3 hand.

18. A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. what of the followings are true?

(a) $a_n = 9a_{n-1} + 10^{n-1}$

(b) $a_n = 8a_{n-1} + 10^{n-1}$

(c) $a_1 = 9$

(d) $a_n = 8^n + 10^{n-1}$

(e) $a_n = 8^n + 10^n$

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19. Find the vertex connectivity and the edge connectivity in the following graph.

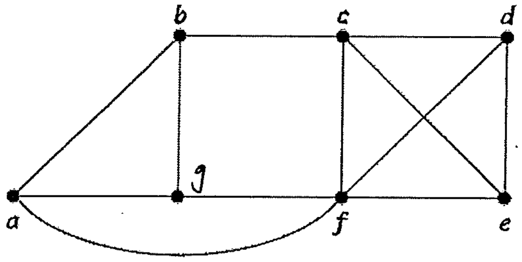
(a) The vertex connectivity is 3.

(b) The vertex connectivity is 2.

(c) The edge connectivity is 3.

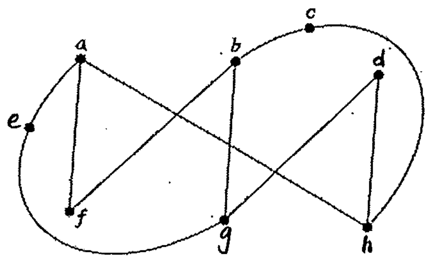
(d) The edge connectivity is 2.

(e) The edge connectivity is 1.

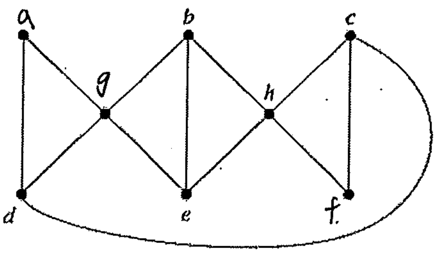


20. Which of these graphs are homeomorphic to $K_{3,3}$?

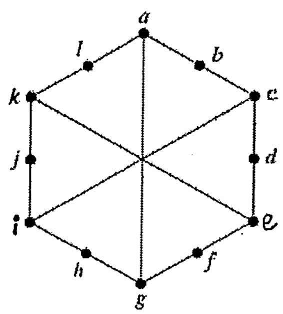
(a)



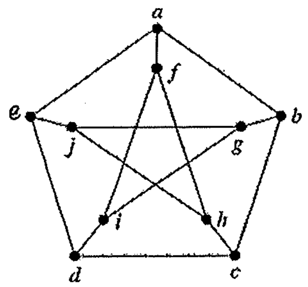
(b)



(c)



(d)



(e)