所別：資訊工程學系碩士班
軟體工程研究所碩士班　　科目：離散數學與線性代數

※請務必按照題號次序寫在答案紙上，否則將嚴重失分。

1. (10%) Suppose x, y represents people. Let $L(x, y)$ be the predicate of "x likes y," $F(x, y)$ be the predicate of "x is a friend of y," $H(y)$ be the predicate of "y is a happy person." Choose the correct logic statement(s) which have the same meaning as the sentence --- **A happy person is a person liked by some of his friends**.

   (a) (4%) (single or multiple choice, no points will be awarded if only partially correct)

   (i) $\forall x\,(\forall y,\,(L(y, x) \wedge F(x, y)) \rightarrow H(x))$.
   (ii) $\exists x\,(\forall y,\,(L(x, y) \wedge F(y, x)) \rightarrow H(x))$.
   (iii) $\forall x\,(\forall y,\,(L(y, x) \rightarrow (\neg F(x, y)) \rightarrow \neg H(x))$.
   (iv) $\forall x\,(\exists y,\,(L(y, x) \wedge F(y, x)) \rightarrow H(x))$.
   (v) none of the above.

   (b) (6%) Suppose there are 10 persons in the universe of discourse, all are friends of each other. Assume every person likes at least 5 other persons in this group, derive a close upper bound of the number of **unhappy** person according to the definition of happy person in (a).

2. (15%) Suppose there is an evaluation process $\boldsymbol{E}$ used to select a set of committee from a set of candidates.

   Process $\boldsymbol{E}$ will first checks (1) if the number of candidates is 2. If so then make 1 comparison step to qualify 1 committee member. (2) Else if the number of candidates is 1, then do 1 qualification step to qualify that candidate. (3) Else if the number of input candidates is greater than 2, randomly select $\lceil 1/3 \rceil$ of them and initiate the process $\boldsymbol{E}$ for the reduced candidate set; randomly select another $\lceil 1/3 \rceil$ of the input candidates and feed them into another process $\boldsymbol{E}$ of evaluation. (random selection for reduced candidate set does not cost anything in terms of the complexity we consider)

   (a) (7%) Suppose each comparison or qualification counts for 1 step, and the number of initial candidates is $\boldsymbol{n}$, derive a recurrence definition for the *number of steps* $\boldsymbol{S(n)}$ that process $\boldsymbol{E}$ takes to evaluate all $\boldsymbol{n}$ candidates.

   (b) (8%) Estimate the complexity order (O(...)) of $\boldsymbol{S(n)}$.

(還有第二頁)

注意：背面有試題

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3. (15%) Let A be a set and let $f: A \to A$ be a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$.
   (a) (5%) Prove that $\sim$ is an equivalence relation on A.
   (b) (5%) For $A = R$ and $f(x) = \lfloor x \rfloor$, find the equivalence classes of $0, \frac{7}{5}, -\frac{9}{16}$
   (c) (5%) Suppose $A = \{1, 2, 3, 4, 5, 6\}$ and $f = \{(1,2), (2,1), (3,1), (4,5), (5,6), (6,1)\}$. Find all equivalence classes.

4. (10%) Let a be any real number. Show that two sets $(0, 1)$ and (a, ∞) have the same cardinality.

5. (12%) Without multiplying matrices, find the four fundamental subspaces, *i.e.*, column space (image) and null space (kernel) of A and A^T, if you know
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. (13%) Use Cramer's Rule to solve $\boldsymbol{Ax} = \boldsymbol{b}$. Also find the inverse matrix $A^{-1} = C^T/\det A$.
$$\boldsymbol{Ax} = \boldsymbol{b} \text{ is } \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$
Why is the solution $\boldsymbol{x}$ in the first part the same as column 3 of A^{-1}? Which cofactors are involved in computing that column $\boldsymbol{x}$?

7. (10%) (*a*) (7%) Find a real general solution of linear system $\boldsymbol{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \boldsymbol{x}$.
   (*b*) (3%) Describe the shapes of solution trajectory.

8. (10%) (*a*) (7%) Find a singular value decomposition of matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.
   (*b*) (3%) Show that the eigenvalues of $A^T A$ are all nonnegative.

9. (5%) If matrix A has linearly independent columns and can be factorized into $\boldsymbol{QR}$, where $\boldsymbol{Q}$ has orthonormal columns, show that $\boldsymbol{R}$ is an upper triangular matrix with positive entries on its diagonal.

(後面沒有題目了)