所別：通訊工程學系碩士班 通訊系統與訊號處理組 科目：工程數學

1.(13%) Let V denote the inner product space of real-valued continuous functions on $[0,T]$. For $r(t), s_1(t), s_2(t), \cdots, s_M(t) \in V$, we want to find $s_i(t)$ $(1 \le i \le M)$ which has the minimum value of $\|r(t) - s_i(t)\|$, the distance between $r(t)$ and $s_i(t)$.

(a)(6%) Suppose that $\|s_1(t)\| = \|s_2(t)\| = \cdots = \|s_M(t)\|$. Prove that $s_i(t)$ which minimizes $\|r(t) - s_i(t)\|$ also maximizes the inner product $\langle r(t), s_i(t)\rangle$.

(b)(7%) Let $\beta = \{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$ represent an ordered orthonormal basis for V. Let the coordinate vector of $r(t)$ and $s_i(t)$ relative to β be $[r(t)]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $[s_i(t)]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, respectively. Prove that $\|r(t) - s_i(t)\|^2 = \sum_{j=1}^{n} |a_j - b_j|^2$.

2.(14%) Let $P(R)$ denote the vector space which consists of all polynomials with coefficients from R, the field of real numbers. Let W be the set of all polynomials $f(x)$ in $P(R)$ such that in the representation $f(x) = a_{4n-1}x^{4n-1} + a_{4n-2}x^{4n-2} + a_{4n-3}x^{4n-3} + \cdots + a_1 x + a_0$ where n is a positive integer, we have $a_i = 0$ whenever i is even (including 0) and $\sum_{j=1}^{n} a_{4j-1} = \sum_{j=1}^{n} a_{4j-3} = 0$.

(a)(7%) Is W a subspace of $P(R)$? Justify your answer.

(b)(7%) If the answer for (a) is "Yes", find a basis for W; otherwise find the smallest subspace that contains W.

3.(10%) Recall the definition of $P(R)$ in the previous question. Define $T: P(R) \to P(R)$ by $T(f(x)) = \int_0^x f(t)dt$ and define $U: P(R) \to P(R)$ by $U(f(x)) = f'(x)$. Determine whether UT (first T then U) and TU are one-to-one or onto. Justify your answer.

4. (a)(6%) Evaluate the determinant of $\begin{bmatrix} 6 & 1 & -1 & 5 \\ 2 & -1 & 3 & -2 \\ 1 & 0 & -1 & 0 \\ -4 & 3 & 2 & 1 \end{bmatrix}$.

(b)(7%) Find all solutions to the system $2x_1 - x_2 + x_3 = 2$ and $x_1 + 3x_2 - x_3 = 1$.

5. (12%) Let $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$. Show that

$$\Gamma(n + 1/2) = \frac{(2n)!}{n! 2^{2n}} \sqrt{\pi}, \qquad n = 0,1,2,....$$

6. (13%) Let X have the beta distribution, which has a probability density function

$$f_X(x) = Cx^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1,$$

where C is the normalization constant. Determine the probability density function of the random variable $X^{-1} - 1$.

注意：背面有試題

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7. (10%) A quiz was administered to four students. Somehow the quizzes got shuffled, and the one at the top of the stack was returned to the first student, the one below it was returned to the second student, and so on. Find the probability that at least one student got his own quiz back.

8. (15%) An urn contains N balls, identical in every respect except that they carry numbers $(1,2,\ldots,N)$ and M of them are colored red, the remaining $(N-M)$ white, $0 \leq M \leq N$. We draw a ball from the urn blindfolded, observe and record its color, lay it aside, and repeat the process until n balls has been drawn, $0 \leq n \leq N$.

(a) (7%) Find the probability of red on the first r consecutive draws in a specified order.

(b) (8%) Find the probability of red on the third draw.