國立中央大學113學年度碩士班考試入學試題

所別：數學系碩士班 數學組(一般生)

科目：高等微積分

*本科考試禁用計算器

以下全部試題，包含子題，皆為計算或證明題。作答需給出清晰，準確之論述與計算過程，論述之完整性將納入評分。若無任何計算過程，該題將不予計分。

Problem 1. We consider the function $f : x \mapsto \dfrac{\ln x}{x-1}$ defined for all $x \in (0,1) \subset \mathbb{R}$.

1.) (5 points) For all $n \in \mathbb{N}$, calculate $\int_0^1 x^n \ln x dx$.

2.) (5 points) Calculate $\lim_{x \to 1} f(x)$.

3.) (10 points) Show that f is integrable on $(0,1)$ and calculate $\int_0^1 f(x)dx$.

Problem 2. Let $n \in \mathbb{N}$.

- $\mathcal{M}_n(\mathbb{R})$ denotes the vector space of $n \times n$ matrices with real coefficients.
- For all $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $||x|| = \sqrt{\sum_{i=1}^{n} x_i^2}$.
- For all $A \in \mathcal{M}_n(\mathbb{R})$, we define

$$||A||_M := \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{||Ax||}{||x||}.$$

1.) (5 points) Consider the particular case $\tilde{A} = \begin{pmatrix} -3 & 1 \\ 0 & 2 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. Calculate $||\tilde{A}||_M$.

2.) (5 points) Following the previous question, calculate $\sup_{\substack{y \in \mathbb{R}^2 \\ ||y||=1}} ||\tilde{A}y||^2$ by using the Lagrange multipliers.

In the next questions, we consider the general cases $A \in \mathcal{M}_n(\mathbb{R})$.

3.) (5 points) Show that $||A||_M = \sup_{\substack{y \in \mathbb{R}^n \\ ||y||=1}} ||Ay||$.

4.) (5 points) Show that for all matrices $B \in \mathcal{M}_n(\mathbb{R})$, $||AB||_M \leq ||A||_M ||B||_M$.

5.) (10 points) Show that the set of invertible matrices $GL(n)$ is an open set in $\mathcal{M}_n(\mathbb{R})$.

注意：背面有試題

Problem 3. Let $(E, \|\cdot\|)$ be a normed space. We recall that a series $\sum_n x_n$ is said to be absolutely convergent if the numerical series $\sum_n \|x_n\|$ is convergent.

1.) (5 points) Is the series $\sum_n \frac{1}{n}$ convergent? Justify your answer and calculate its sum if you have a positive answer.

2.) (5 points) Is the series $\sum_n \frac{(-1)^{n-1}}{n}$ convergent? Justify your answer and calculate its sum if you have a positive answer.

3.) (8 points) Show that if $(E, \|\cdot\|)$ is complete, then any absolutely convergent series converges in E.

4.) (6 points) Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(E, \|\cdot\|)$. Show that there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$, $\|y_{n_k} - y_{n_{k+1}}\| \leq 2^{-k}$.

5.) (6 points) We assume that any absolutely convergent series converges in E. Show that $(E, \|\cdot\|)$ is complete.

Problem 4. Let $a, b, K \in \mathbb{R}$ with $a < b$ and $K > 0$. We say a real-valued function g defined on $[a, b]$ is K-Lipschitz continuous if

$$\forall x, y \in [a, b],\ |g(x) - g(y)| \leq K|x - y|.$$

Consider the sequence of real-valued functions $(f_n)_{n\in\mathbb{N}}$ defined on $[a, b]$. We assume that for all $n \in \mathbb{N}$, f_n is K-Lipschitz continuous and the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwisely to f on $[a, b]$. That means,

$$\forall x \in [a, b],\ f(x) = \lim_{n\to+\infty} f_n(x).$$

1.) (5 points) Show that the function f is K-Lipschitz continuous on $[a, b]$.

2.) (15 points) Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$.

試題結束

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