

國立中央大學 114 學年度碩士班考試入學試題

系所： 數學系 碩士班 數學組(一般生)

第 1 頁 / 共 1 頁

科目： 高等微積分

*本科考試禁用計算器

以下全部試題，包含子題，皆為證明題。作答需給出清晰，準確之論述與證明過程，論述之完整性將納入評分。

Let \mathbb{N} be the natural numbers and \mathbb{R} be the set of real numbers.

1. (15 points) We say that $A \subseteq \mathbb{R}^n$ is compact if every open cover of A has finite subcover. Show that every compact set in \mathbb{R}^n is closed by the definition.
2. (15 points) Let K be compact metric space and f_n be continuous function on K for all $n \in \mathbb{N}$. If $\{f_n\}$ converges uniformly on K , show that $\{f_n\}$ is equicontinuous on K . That is, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $d(x, y) < \delta$, $x, y \in K$, and $n \in \mathbb{N}$.
3. (15 points) Prove that a polynomial of degree n is uniformly continuous on \mathbb{R} if and only if $n = 0$ or 1 .
4. (15 points) Suppose that f is differentiable on \mathbb{R} . If $f(0) = 2$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \leq |x| + 2$ for all $x \in \mathbb{R}$.
5. (20 points) For $0 < x < \infty$, define

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Prove that (a) (10 points) the integral converges for $0 < x < \infty$;

(b) (10 points) the equation $\Gamma(x+1) = x\Gamma(x)$ holds if $0 < x < \infty$.

6. (20 points) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Investigate the continuity and differentiability of f at $(0, 0)$.