參考用

1. (*20* points) The following computer statements are written in pseudocodes similar to the C programming language.
1.1 Let n be claimed as an integer in the C (or FORTRAN) programming language. Consider the following statements. When they are executed by a computer, will the execution eventually stop? How and why?

```
n = 1;
while (n > 0)
        n = 2 * n;
```

1.2 Let x and y be claimed as floating numbers. Consider the following statements. Please explain what is the purpose for these statements; that is, mathematically, what will be the value of y as a function of x?

```
y = 1;
for i = 1, 2, ···, 10
        x = x/i;
        y = y + x;
```

2. (*20* points) Let $w(x)$ be a positive weight function which is integrable on the interval $[a,b]$. Let $q(x)$ be a polynomial of degree $n+1$ which is w-orthogonal to Π_n; that is,

$$\int_a^b w(x)p(x)q(x)\,dx = 0$$

for all polynomials $p(x)$ of degree $\leq n$. Suppose $q(x)$ has $n+1$ distinct zeros in (a,b), say, $x_0, x_1, \ldots, x_n$.
2.1 Explain how to find quadrature parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that the numerical quadrature rule

$$\int_a^b f(x)w(x)\,dx \approx \sum_{i=0}^{n} \alpha_i f(x_i)$$

is *exact* for all polynomials $f(x)$ of degree $\leq n$. That is, $\approx$ can be replaced by $=$.
2.2 Prove that the numerical quadrature rule you designed in the previous problem is actually *exact* for any polynomial of degree $\leq 2n+1$. This is the generalized Gaussian quadrature rule.

3. (*20* points) The B-spline of degree 0 on integer nodal points is defined by

$$B^0(x) = \begin{cases} 1 & \text{for } x \in [0,1), \\ 0 & \text{otherwise.} \end{cases}$$

The B-spline of degree j on integer nodal points are recursively defined by

$$B^j(x) = \frac{x}{j}B^{j-1}(x) + \frac{j+1-x}{j}B^{j-1}(x-1), \quad j > 0.$$

3.1 Use the recursion relation, write down the formulae for $B^1(x)$ and $B^2(x)$.

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3.2 Prove that, for any $j > 0$,

$$\sum_{k=-\infty}^{\infty} B^j(x-k) = \sum_{k=-\infty}^{\infty} B^{j-1}(x-k).$$

Hence

$$\sum_{k=-\infty}^{\infty} B^j(x-k) = 1.$$

4. (*20* points) Let A be a real-valued matrix of order $m \times n$, and x, b be real-valued (column) vectors of dimension n and m respectively. Assume that $m > n$ and rank$A = n$.
4.1 What is the sufficient and necessary condition that the system of equations

$$Ax = b \tag{1}$$

has a unique solution? Just state it, you do not have to prove.
4.2 In general there will be no solution for equation (1). But there is a unique solution such that $\|Ax - b\|_2$ is minimal. Here $\|\cdot\|_2$ is the vector ℓ^2-norm. Such a solution is called the *least square* solution. Prove that, the normal equation

$$A^T Ax = A^T b$$

has a unique solution x, and this x is the least square solution of (1). [Hint. $Ax - b$ is orthogonal to the range of the linear operator A. Then show that $\|Ay - b\|_2 \geq \|Ax - b\|_2$, for any n-vector y.]

5. (*20* points) Let $\phi(x)$ be a continuously differentiable function with support on $[0, N]$, where N is a positive integer. Suppose $\phi(x)$ satisfies the equation

$$\sum_{k=-\infty}^{\infty} k\phi(x-k) = x + c$$

for certain constant c. Let

$$M_k = \int_{-\infty}^{\infty} \phi(x)\phi'(x-k)\,dx,$$

where k is any integer.
5.1 Show that $M_k = 0$ for $|k| \geq N$.
5.2 Show that $M_0 = 0$.
5.3 Show that $M_{-k} = -M_k$.
5.4 What is the value of

$$\sum_{k=-\infty}^{\infty} kM_k?$$