1. Prove that for any real symmetric matrix A, there is an orthogonal matrix U and a diagonal matrix D such that $A = U \cdot D \cdot U^t$. (20%)

2. Let A be a real m by n matrix. Prove or disprove that (20%)

a. $\text{rank}(A) = \text{rank}(A^t \cdot A)$.

b. $\text{rank}(A \cdot A^t) = \text{rank}(A^t \cdot A)$.

c. Both $A \cdot A^t$ and $A^t \cdot A$ share the same set of non-negative eigen values.

d. Both $A \cdot A^t$ and $A^t \cdot A$ share the same set of eigen vectors.

3. Let A be a real m by n matrix (20%)

a. Prove that $f(x) = A \cdot x$ define a linear isomorphism between the row space and column space of A.

b. Find an orthonormal basis $\{u_1, u_2, ..., u_r\}$ of the column space, and an orthonormal basis $\{v_1, v_2, ..., v_r\}$ of the row space such that $A \cdot v_i = c_i \cdot u_i$ for some c_i, i = 1,2 ...,r.

4. Let $A = (a_{i,j})$ be a 10 by 10 square matrix defined by $a_{i,i} = 2$ for all i=1,...,10, $a_{i,j} = 1$ for all i $\neq$ j (20%)

a. Prove that A is positive definite.

b. Find all eigen values and corresponding eigen vectors.

5. Let $A = \begin{pmatrix} 4 & 16 & -14 \\ 16 & 10 & -2 \\ -14 & -2 & -5 \end{pmatrix}$ (20%)

a. What is rank(A),trace, determinant of A ?

b. Determine whether $4x^2 + 10y^2 + 32xy - 28x - 4y - 5$ is factorizable over real field.