所別：數學系碩士班 不分組 科目：高等微積分

R denotes the set of all real numbers.

1. Let $a \in R$ and define the sequence $a_1, a_2, \ldots$ in R by $a_1 = a$, and $a_n = a_{n-1}^2 - a_{n-1} + 1$ if $n > 1$. For what $a \in R$ is the sequence $\{a_n\}$
   (a) Monotone ? (3%)
   (b) Bounded ? (3%)
   (c) Convergent ? Compute the limit in the cases of convergence. (4%)

2. (20%) Let $C = \{\sum_{n=1}^{\infty} \frac{a_n}{3^n} | a_n = 0 \text{ or } 2 \text{ for each } n\}$. Prove :
   (a) C is a **compact set** in R.
   (b) C is **uncountable**.
   (c) $\text{int}(C) = \emptyset$ (empty set).
   (d) Show that C is **totally disconnected**; that is, if $x, y \in C$ and $x \neq y$ then $x \in U$ and $y \in V$ where U and V are open sets that disconnect C.

3. (15%) Let f be a continuous function on $[0, \infty)$ such that $0 \leq f(x) \leq Cx^{-1-\varepsilon}$ for some $C, \varepsilon > 0$, and let $A = \int_0^\infty f(x)dx$. Let $f_n(x) = nf(nx)$.
   (a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$ and that the convergence is uniform on $[\delta, \infty)$ for any $\delta > 0$.
   (b) Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = A$.
   (c) Show that $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = Ag(0)$ for any integrable function g on $[0, 1]$ that is continuous at 0. (Hint: Write $\int_0^1 = \int_0^\delta + \int_\delta^1$.)

4. Let $f_n(x) = xe^{-nx}, x \in [0, \infty), n = 0, 1, 2, \ldots$.
   (a) Show that $f(x) = \sum_{n=0}^{\infty} f_n(x)$ exists. Compute f explicitly. (2%)
   (b) Is f continuous ? (3%)
   (c) Find a suitable set on which the convergence is uniform.(5%)
   (d) May we differentiate term by term on $(0, \infty)$? Why ? (5%)

5. (20%) Show that if $f : A \subset R^2 \to R$ has a critical point $(x_0, y_0) \in A$ and we let
$$\triangle = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
   be evaluated at (x_0, y_0), then
   (a) $\triangle > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$ imply f has a local minimum at (x_0, y_0).
   (b) $\triangle > 0$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0)) < 0$ imply f has a local maximum at (x_0, y_0).
   (c) $\triangle < 0$ implies (x_0, y_0) is a saddle point of f.
   (d) Determine the nature of the critical points of $f(x, y) = x^3 + y^2 - 6xy + 6x + 3y$.

6. (20%)
   (a) Can the equation $\sqrt{x^2 + y^2 + 2z^2} = \cos z$ be solved uniquely for y in terms of x and z near $(0, 1, 0)$? For z in terms of x and y ?
   (b) Investigate the possibility of solving the equations $u^3 + xv - y = 0, v^3 + yu - x = 0$ for any two of the variables as functions of the other two near the point $(x, y, u, v) = (0, 1, 1, -1)$.