所別：數學系碩士班 甲組　　科目：線性代數

(1) (10%) Prove that the space of all real $n \times n$ matrices is a direct sum of the subspace S_1 of symmetric matrices and the subspace S_2 of skew-symmetric matrices. Find the projections A_1 and A_2 of the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

on S_1 along S_2 and on S_2 along S_1.

(2) (10%) Prove that for an $m \times n$ matrix A to have rank 1, it is necessary and sufficient for A to be represented as $A = BC$ where B is a nonzero $m \times 1$ matrix and C is a nonzero $1 \times n$ matrix.

(3) (13%) Let $A = \begin{bmatrix} 3 & 1 & 3 & 1 \\ -3 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Find a vector x_0 satisfies the following conditions:

(a) $\|A \cdot x_0 - b\| \le \|A \cdot x - b\|$ for all x,

(b) Find x_0 satisfies (a) with minimal norm.

(4) (12%) Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$

(a) Prove that $\max_{\|x\|=1} \|A \cdot x\|$ is well defined,

(b) Find $\max_{\|x\|=1} \|A \cdot x\|$, $\min_{\|x\|=1} \|A \cdot x\|$.

(5) (10%) Let A be a real n by n matrix. Prove or disprove that if $A^5 - 3A^3 + 2A = 0$ then A is diagonalizable over real field.

(6) (15%) Let

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

(a) Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}BQ$.

(b) Find all the possible Jordan forms for a matrix which has the same characteristic polynomial of B.

(7) (15%) Let A= $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ Prove or disprove that

(a) There is a real 5×5 lower triangular matrix B such that $A = B \cdot B^t$,

(b) There is a real 5×5 matrix C such that $A = C^2$.

(8) (15%) Let $P_3(\mathbb{R})$ be the space of all real polynomials having degree less than or equal to 3 with the inner product $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)dt$.

(a) Use the Gram-Schmidt process to replace $\beta = \{1, 1+x, x+x^2, x^2+x^3\}$ by an orthonormal basis for $P_3(\mathbb{R})$.

(b) Let T be a linear operator on $P_3(\mathbb{R})$ defined by $T(f(t)) = f'(t) + 3f(t)$. Find the matrix representation of the adjoint T^* of T in the orthonormal basis found in (a).