國立中央大學114學年度碩士班考試入學試題

系所： 統計研究所 碩士班 不分組(一般生)
統計研究所 碩士班 不分組(在職生)

科目： 基礎數學

＊本科考試可使用計算器，廠牌、功能不拘

計算題應詳列計算過程，無計算過程者不予計分

1. (16%)

   (a) (8%) Let $g(x)$ be a nondecreasing nonnegative function for $0 < x < \infty$ and $0 \le f(x) \le 1$ for $0 < x < \infty$. Given $g(\epsilon) > 0$ with some $\epsilon > 0$, show that

   $$\int_{g(\epsilon)}^{\infty} f(y)dy \le \frac{1}{g(\epsilon)} \int_{-\infty}^{\infty} g(y)f(y)dy.$$

   (b) (8%) Let α be a positive integer and $\lambda > 0$. Show that

   $$\int_{\epsilon^2}^{\infty} y^{\alpha-1}e^{-\lambda y}dy \le \frac{1}{\epsilon^2}\frac{(\alpha+1)!}{\lambda^{\alpha+2}}.$$

2. (24%)

   (a) (8%) Show that

   $$\int_0^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{2}.$$

   (b) (8%) Show that for all $y \ge 0$

   $$\int_y^{\infty} e^{-\frac{x^2}{2}}dx \le \frac{1}{y}e^{-\frac{y^2}{2}}.$$

   (c) (8%) Show that

   $$\int_1^{\infty}\int_y^{\infty} e^{-\frac{(x^2+y^2)}{2}}dxdy \le \sqrt{\pi}.$$

3. (16%) Give constants $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $0 < \mu < 1$. Minimize $\beta_1 w_1^2 + \beta_2 w_2^2$ with respect to w_1 and w_2 under two constraints $w_1 + w_2 = 1$ and $\alpha_1 w_1 + \alpha_2 w_2 = \mu$ with $0 \le w_1 \le 1$ and $0 \le w_2 \le 1$.

   (a) (8%) Find the minimizer w_1 and w_2.

   (b) (8%) Find the minimal value of $\beta_1 w_1^2 + \beta_2 w_2^2$.

4. (16%) Given the two by two matrix

   $$G = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

   with $\lambda > 0$ and $\mu > 0$.

   (a) (8%) Find the eigendecomposition $G = B\Lambda B^{-1}$.

   (b) (8%) Evaluate $e^G = \sum_{n=0}^{\infty} \frac{1}{n!}G^n$ using $G = B\Lambda B^{-1}$. Note that $G^0 = I$ where I is the identity matrix.

注意:背面有試題

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5. (28%) Give a full rank matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{pmatrix},$$

and $H = A(A^TA)^{-1}A^T$ where A^T is the transpose of A and A^{-1} is the inverse of A.

(a) (7%) Show that H^5 is a positive semi-definite matrix.

(b) (7%) If $rank[H^5] = r \leq 5$, show that it has r eigenvalues equal to unity and $5 - r$ eigenvalues equal to zero.

(c) (7%) Show $trace[H^5] = rank[H^5]$.

(d) (7%) Find the eigenvalues for $I_5 - H^5$ where I_5 is the 5×5 identity matrix.

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