國立中央大學九十一學年度碩士班研究生入學試題卷

所別： 統計研究所 不分組 科目： 基礎數學 共 1 頁 第 1 頁

1. Let the real-valued function f be defined on $(-\infty, \infty)$ by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

   (a) Prove or disprove that f is differentiable in $(-\infty, \infty)$. (8%)

   (b) Prove or disprove that the first derivative function f' is a continuous function. (8%)

2. Find $\lim_{x \to 0} \dfrac{e - (1+x)^{\frac{1}{x}}}{2x}$. (10%)

3. (a) Prove that if $x > 0$ and $\alpha > 0$, then $\lim_{n \to \infty} \dfrac{n^\alpha}{(1+x)^n} = 0$. (8%)

   (b) Let $f_n(x) = nx(1-x^2)^n$, $0 \le x \le 1$, $n = 1, 2, \cdots$. Prove or disprove

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = \int_0^1 \left[\lim_{n \to \infty} f_n(x)\right] dx \; .$$

(8%)

4. (a) Directly compute $\int_0^\infty \int_0^\infty e^{\frac{-(x^2+y^2)}{2}} dx\, dy$ without quoting from any formula. (8%)

   (b) Using the result of (a), compute $\int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$. (8%)

5. Let $\boldsymbol{A}$ be an $m \times n$ matrix $(m \neq n)$.

   (a) Prove that if $tr(\boldsymbol{A}'\boldsymbol{A}) = 0$, then $\boldsymbol{A} = \boldsymbol{0}$. (6%)

   (b) Prove or disprove that if $m = n$ and $tr(\boldsymbol{A}^2) = 0$, then $\boldsymbol{A} = \boldsymbol{0}$. (6%)

6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ symmetric matrices, and let $\boldsymbol{AB}$ be idempotent. Show that $\boldsymbol{BA}$ is also idempotent. (10%)

7. Let $\boldsymbol{A}$ be an $n \times n$ positive definite matrix, and let $\boldsymbol{B}$ be an $n \times n$ nonnegative definite matrix. Show that $|\boldsymbol{A} + \boldsymbol{B}| \ge |\boldsymbol{A}|$. (10%)

8. Suppose $\boldsymbol{A}$ is an $n \times n$ matrix, and $\boldsymbol{A} = \boldsymbol{I} - \frac{1}{n} \cdot \boldsymbol{1} \cdot \boldsymbol{1}'$, where $\boldsymbol{I}$ is an $n \times n$ identity matrix and $\boldsymbol{1}$ is an n-component column vector with all elements equal to 1. Show that the rank of $\boldsymbol{A}$ is equal to $n - 1$. (10%)

※ Note: 1. All elements of matrices in this exam are real-valued.

2. You may use the following definition to answer the questions.

   Definiton: Let $\boldsymbol{A} = (a_{ij})$, $1 \le i \le n$, $1 \le j \le n$.

   (1) The trace of $\boldsymbol{A}$ is defined as $tr(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}$.

   (2) $\boldsymbol{A}$ is called idempotent if $\boldsymbol{A}^2 = \boldsymbol{A}$.

   (3) $\boldsymbol{A}$ is said to be nonnegative definite if for every real vector $\boldsymbol{x} \neq 0$, $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \ge 0$, and $\boldsymbol{A}$ is said to be positive definite if for every real vector $\boldsymbol{x} \neq 0$, $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0$.