國立中央大學99學年度碩士班考試入學試題卷

所別：統計研究所碩士班 不分組(一般生) 不分組(在職生)　科目：基礎數學　共 1 頁 第 1 頁

＊請在試卷答案卷（卡）內作答

＊本科考試可使用計算器，廠牌、功能不拘

1. Assume f has a finite derivative in an open interval (a, b) and is continuous at both endpoints a and b with $f(a) = f(b) = 0$. Prove that for every real value θ there is some c in (a, b) such that $f'(c) = \theta f(c)$. *(10%)*

2. For all $x > 0$, prove that

   (i) $\int_x^\infty e^{-\frac{t^2}{2}} dt < \frac{1}{x} e^{-\frac{x^2}{2}}$. *(6%)*

   (ii) $\frac{x}{1+x^2} e^{-\frac{x^2}{2}} < \int_x^\infty e^{-\frac{t^2}{2}} dt$. *(6%)*

3. Let $a_n = n^2 \sin^2\left(\frac{n\pi}{3}\right)$. Find $\lim_{n\to\infty} \inf a_n$ and $\lim_{n\to\infty} \sup a_n$. *(10%)*

4. Assume that each $a_n > 0$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\prod_{n=1}^{\infty}(1 + a_n)$ converges. *(5+5=10%)*

5. Let $\{f_n\}$ be a sequence of functions which converges pointwise on a set S. Assume $\{f_n\}$ is uniformly bounded on S. Does $\{f_n\}$ contain a uniformly convergent subsequence on S? Prove or disprove your answer. *(3+10=13%)*

6. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. Let $r(\mathbf{A})$ denote the rank of $\mathbf{A}$. Show that $r(\mathbf{AB}) \geq r(\mathbf{A}) + r(\mathbf{B}) - n$. *(10%)*

7. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be $n \times n$ symmetric matrices such that $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{I}$. If $\mathbf{A}^2 = \mathbf{A}$, $\mathbf{B}^2 = \mathbf{B}$, and $\mathbf{C}$ is nonnegative. Show that $\mathbf{C}^2 = \mathbf{C}$. *(10%)*

8. Suppose $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices and $\mathbf{A} = \mathbf{B}$. Prove that there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P}$ and $\mathbf{P}'\mathbf{B}\mathbf{P}$ are both diagonal. *(10%)*

9. Let $\mathbf{A}$ be an $n \times m$ matrix. An $m \times n$ matrix $\mathbf{A}^*$ is said to be a generalized inverse (g-inverse) of $\mathbf{A}$ if $\mathbf{AA}^*\mathbf{A} = \mathbf{A}$.

   (a) Prove that $r(\mathbf{A}) = r(\mathbf{AA}^*) = r(\mathbf{A}^*\mathbf{A})$. *(5%)*

   (b) Suppose $\mathbf{A}$ is an $n \times m$ matrix of rank $r < \min(n, m)$. (i) Find the g-inverse of $\mathbf{A}$. (ii) Prove or disprove the g-inverse of $\mathbf{A}$ is not unique. *(5+5=10%)*