國立中央大學104學年度碩士班考試入學試題

所別：天文研究所碩士班 不分組(一般生)　　科目：應用數學　　共 2 頁　第 1 頁
天文研究所碩士班 不分組(在職生)

本科考試禁用計算器　　　　＊請在答案卷（卡）內作答

1. (total 20%) Find the limits

   (i) (4%) $\lim_{x\to\infty}\frac{2x^2-5x+2}{5x^2-7x-6}$　(ii) (4%) $\lim_{x\to 2}\frac{2x^2-5x+2}{5x^2-7x-6}$

   (iii) (4%) $\lim_{x\to 0}\frac{\cosh x-1}{1-\cos 2x}$

   (iv) (4%) $\lim_{x\to\infty} x^n e^{-x}$, where n is a finite positive integer

   (v) (4%) $\lim_{x\to 0^+} x^x$

2. (total 15%) If a curve is represented by a continuously differentiable function $\vec{r}=\vec{r}(t)$

   (i) (5%) Show that the length of the curve l between $t=a$ and $t=b$ where $b>a$ is $l=\int_a^b\sqrt{\dot{\vec{r}}\cdot\dot{\vec{r}}}\,dt$ where $\dot{\vec{r}}=\frac{d\vec{r}}{dt}$

   (ii) (5%) If the curve is on the x-y plane and y can be written as a function of x, using the equation in (i), show that the length of the curve between $x=x_1$ and $x=x_2$ where $x_2>x_1$ can be written as

   $$l=\int_{x_1}^{x_2}\sqrt{1+(y')^2}\,dx,\text{ where } y'=\frac{dy}{dx}$$

   (iii) (5%) If the curve is represented in polar coordinate (ρ,θ) and the radial variable ρ can be written as a function of angular variable θ, that is $\rho=\rho(\theta)$, using the equation in (i), show that the length of the curve between $\theta=\theta_1$ and $\theta=\theta_2$ where $\theta_2>\theta_1$ can be written as

   $$l=\int_{\theta_1}^{\theta_2}\sqrt{(\rho)^2+\left(\frac{d\rho}{d\theta}\right)^2}\,d\theta$$

3. (total 15%) The Gaussian distribution is described as

   $P(x)=A\cdot\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$ where A is the normalization constant and $-\infty<x<\infty$

   (i) (5%) Find the normalization constant A so that $\int_{-\infty}^{\infty}P(x)\,dx=1$

注意：背面有試題

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參考用

The expectation value of a function $f(x)$ is defined as

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)P(x)dx$$

(ii) (5%) calculate the mean: $\langle x\rangle$ and

(iii) (5%) variance: $\left\langle \left(x-\langle x\rangle\right)^2\right\rangle$.

4. (total 15%) For an $n\times n$ matrix $\mathbf{A}$ whose matrix elements are $[\mathbf{A}]_{ij} = a_{ij}$, its adjoint matrix is defined as $\mathbf{A}^\dagger$ whose matrix elements are $\left[\mathbf{A}^\dagger\right]_{ij} = a_{ji}^*$ where a_{ji}^* is the complex conjugate of a_{ji}

   (i) (5%) show that $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$

   (ii) (5%) show that $trace\left(\mathbf{A}^\dagger\mathbf{A}\right)\geq 0$

   (iii) (5%) $\mathbf{A}$ is Hermitian matrix if $\mathbf{A}^\dagger = \mathbf{A}$. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian matrices and $\mathbf{AB}-\mathbf{BA} = i\mathbf{C}$, show that $\mathbf{C}$ is a Hermitian matrix

5. (total 20%) A triangular wave is given by

$$f(x) = \begin{cases} x & \text{for } 0<x<\pi \\ -x & \text{for } -\pi<x<0 \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$

   (i) (10%) Represent $f(x)$ by a Fourier series

   (ii) (10%) Use the result from (i) to calculate $\sum_{n=0}^{\infty}\frac{1}{(2n+1)}$

6. (15%) Show that

$$\sum_{n=0}^{N-1}\cos nx = \frac{\sin\frac{Nx}{2}}{\sin\frac{x}{2}}\cos\left[\frac{(N-1)x}{2}\right]$$

$$\sum_{n=0}^{N-1}\sin nx = \frac{\sin\frac{Nx}{2}}{\sin\frac{x}{2}}\sin\left[\frac{(N-1)x}{2}\right]$$

注意：背面有試題