國立中央大學八十四學年度碩士班研究生入學試題卷

PLEASE READ THIS SHORT MESSAGE FIRST:
Please work out the following problems in detail, otherwise put down how you may proceed. Attempt as many problems as you can but spend your time *wisely*. Please pay attention to the score of each problem. Good luck!

(1) (25 points) Consider the three dimensional heat equation

$$\frac{\partial T}{\partial t} = K\nabla^2 T$$

What is K usually called and what is the dimension (or units) of K?
The cooling of a sphere of radius R is governed by the heat equation above. The surface of the sphere is kept at $T = 0$. If the initial temperature distribution of the sphere is $T(t,r) = T_o R\sin(\pi r/R)/r$, what is the temperature distribution at time t? [Hint: Try $T(t,r) = T_1(t)T_2(r)/r$.]

(2) (25 points) What is the name of the following equation?

$$f(x) = g(x) + \lambda\int_0^1 K(x,y)f(y)\,dy$$

where $g(x)$ and $K(x,y)$ are prescribed functions, and $f(x)$ is unknown. Find $f(x)$ if
(a) $g(x) = e^x$ and $K(x,y) = e^{(x-y)}$, and
(b) $g(x) = e^x$ and $K(x,y) = e^{(x-y)}H(x-y)$, where $H(x-y) = 1$ if $x \geq y$ and $H(x-y) = 0$ if $x < y$.

(3) (25 points) A string of uniform linear density ρ is fixed at $x = 0$ and $x = L$. The tension on the string is T. The displacement $y(t,x)$ of the string is governed by

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

What is the name of this equation? Suppose that $y \propto e^{-i\omega t}$, calculate the eigenfrequencies and the normalized eigenfunctions of the string.
Now a small mass is attached to the string at $x = a$ $(0 < a < L)$. Consider the mass as a small perturbation to the above system, compute the first order correction in the lowest frequency.
[Hint: If a linear differential operator $\hat{H}$ can be written as a sum of an unperturbed part $\hat{H}_0$ and a perturbation $\hat{H}_1$, then the first order correction to the nth unperturbed eigenvalue is given by $\int u_n^* \hat{H}_1 u_n dx$, where u_n is the nth normalized eigenfunction of $\hat{H}_0$.]

(4) (25 points) Consider an orthogonal curvilinear coordinate system (ξ,η,ζ). The element of the arc length ds and the Laplacian operator ∇^2 are given by

$$ds^2 = h_\xi^2 d\xi^2 + h_\eta^2 d\eta^2 + h_\zeta^2 d\zeta^2$$

$$\nabla^2 = \frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\zeta h_\xi}{h_\eta}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial}{\partial\zeta}\right)\right]$$

What is the meaning of h_ξ, h_η and h_ζ? Suppose (x,y,z) is a Cartesian coordinate system and

$$x = a\cosh\xi\,\sin\eta\,\cos\zeta\,, \quad y = a\cosh\xi\,\sin\eta\,\sin\zeta\,, \quad z = a\sinh\xi\,\cos\eta\,,$$

where $\xi \geq 0$, $0 \leq \eta < \pi$, $0 \leq \zeta < 2\pi$ and a is a constant. Find h_ξ, h_η and h_ζ. Sketch the coordinate curves at $\zeta = 0$ in the Cartesian coordinate system. Label the constant ξ and constant η curves.
A scalar function Φ, depends on ξ only, satisfies the Laplace equation, i.e., $\nabla^2\Phi = 0$. Find the general solution for Φ.