國立中央大學九十一學年度碩士班研究生入學試題卷

所別: 天文研究所 不分組 科目: 應用數學 共 2 頁 第 1 頁

請詳列計算過程否則不予計分

1. (10%) y is a function of x and satisfied:

$$\frac{d^2y}{d^2x} + 16y = 0, \tag{1}$$

$y(\pi) = dy/dx(\pi) = 1$.

Please calculate the solution $y(x)$.

2. Laplace transform and inverse Laplace transform

(a) (5%) Find the Laplace transform of $e^{at}t^2$

(b) (5%) Find the inverse Laplace transform of $\frac{1}{(s+2)(s^2+1)}$

(c) (10%) Find the solution of the given linear system

$$\begin{cases} \frac{d^2x}{dt^2} - 3\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = 0, \\ \frac{dx}{dt} + \frac{dy}{dt} - 2x + y = 0, \\ x(0) = 0, y(0) = -1, x'(0) = 0. \end{cases} \tag{2}$$

3. Consider the following differential equations:

$$\begin{cases} \frac{dx}{dt} = -5x - y \\ \frac{dy}{dt} = 8x + 4y \end{cases} \tag{3}$$

(a) (5%) Find the eigenvalues of

$$\begin{bmatrix} -5 & -1 \\ 8 & 4 \end{bmatrix} \tag{4}$$

(b) (10%) Assume that the two eigenvalues you get are α_1 and α_2, prove that

$$\begin{cases} x(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \\ y(t) = C_3 e^{\alpha_1 t} + C_4 e^{\alpha_2 t} \end{cases} \tag{5}$$

are the general solutions of the differential equations if $C_3 = -8C_1$ and $C_4 = -C_2$.

注意:背面有試題

4. (10%) On the x-y plane, calculate

$$\int_c x^2 y dx - y dy,$$

where c is the straight line between (-2, 3) and (1,1)

5. $\mathbf{F} = -5x\,\mathbf{i} + y\,\mathbf{j} - z\,\mathbf{k}$

(a) (5%) Calculate the Divergent of the vector F, that is, $\nabla \mathbf{F}$

(b) (10%) Prove that

$$\int\int\int_V \nabla \mathbf{F} = \int\int_S \mathbf{F},$$

where V is the volume connected by the points: (0,0,0), (1,0,0), (0,1,0) (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1) and S is the surface of this volume.

6. y is a function of x and satisfied:

$$x^2 \frac{d^2y}{d^2x} - 3x\frac{dy}{dx} + (x^2 - 2)y = 0, \tag{6}$$

Assume that

$$y = \Sigma_{n=0}^{\infty} \quad a_n x^{n+r} \tag{7}$$

(a) (5%) Show that $r = 2 \pm \sqrt{6}$

(b) (5%) $a_1 = a_3 = a_5 = \ldots\ldots = a_{2n+1} = 0$

(c) (5%) Prove that

$$a_n = \frac{-a_{n-2}}{(n+r)(n+r-4)-2} \tag{8}$$

7. Beta Function is defined by:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{9}$$

where $x > 0$, $y > 0$.

(a) (5%) Prove that $B(x,y) = B(y,x)$

(b) (5%) Show that $B(x,y) = \int_0^{\pi/2} 2\sin^{2x-1}(\theta)\cos^{2y-1}(\theta)d\theta$

(Hint: assume $t = \sin^2(\theta)$)

(c) (5%) Prove that