

類組：電機類 科目：工程數學 A(3003)

※請在答案卷內作答

1. (15 pts). Write down your proof as detailed as possible.

Let  $V$  be a finite-dimensional vector space over a field  $F$ . The dual space  $V^*$  of  $V$  is defined as the vector space of linear functionals on  $V$ , i.e.  $V^* = \{f | f : V \rightarrow F\}$ . Let  $T : V \rightarrow V$  be a linear operator.  $T$ 's transpose  $T^t : V^* \rightarrow V^*$  is a linear mapping from  $V^*$  to  $V^*$  defined by  $T^t(g) = gT$  for each  $g \in V^*$ . Let  $V = P(\mathbb{R})$ , the vector space of polynomials over real numbers. For each positive integer  $k$ , define  $\varphi_k : V \rightarrow \mathbb{R}$  by  $\varphi_k(f(x)) = f^{(k)}(0)$ , the  $k$ -th derivative of  $f(x)$  at  $x = 0$ . Let  $\partial : V \rightarrow V$  be the differentiation mapping defined by  $\partial(f(x)) = f'(x)$ . Prove that  $\partial^t \varphi_k = \varphi_{k+1}$ .

2. (15 pts). Let  $V = C^\infty$ , the vector space of all real functions having derivatives of all orders, and let  $y_1, y_2, \dots, y_n$  be some fixed linearly independent functions in  $V$ . Let  $\delta : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$  be an alternating  $n$ -linear function (defined for each  $n \times n$  matrix over  $\mathbb{R}$ ) that is not identically to zero. For each  $y \in V$  and  $t \in \mathbb{R}$ , define  $T(y(t)) \in \mathbb{R}$  as follows.

$$T(y(t)) = \delta \begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}$$

- (a) Prove that  $T : V \rightarrow \mathbb{R}$  is a linear transformation.  
 (b) Prove that the null space of  $T$  satisfies  $N(T) \supset \text{Span}(\{y_1, y_2, \dots, y_n\})$ .
3. (15 pts). Eigenvalues and eigenvectors.
- (a) Let  $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ . Find its eigenvalues  $\lambda_1, \lambda_2 \in \mathbb{C}$ .  
 (b) Continuing from above, find a matrix  $Q \in M_{2 \times 2}(\mathbb{C})$  such that  $Q^{-1}AQ$  is diagonal.  
 (c) Find the minimum positive integer  $n$  such that  $A^n = I$ .

4. (10 pts). Least-square approximation. Let  $f(t)$  be defined as follows,

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < \pi \\ -1, & \text{if } -\pi < t < 0. \end{cases}$$

Also, define

$$g(t) = a \cos t + b \cos 2t + c \sin t.$$

Find the coefficients  $(a, b, c)$  such that  $E = \int_{-\pi}^{\pi} |g(t) - f(t)|^2 dt$  is minimized.

參考用

注意：背面有試題

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5. (25 pts). For a system of non-linear ordinary equations (with  $m > 0$ ) in a two dimension phase plane

$$\begin{aligned}y_1' &= y_2 - 2, \\y_2' &= \frac{2m}{\pi}y_1 - \sin y_1,\end{aligned}$$

- (a) For  $m = 1$ , please find all the critical points in the phase plane.  
(b) Find the range for the value of  $m$  such that this system of ordinary differential equation has seven critical points.

6. (20 pts). Complex integrals

- (a) Find all the poles of

$$\frac{e^{az}}{1+e^z}, \quad \text{where } 0 < a < 1.$$

- (b) Calculate the integral for the contour  $C : \text{Im}(z) > 0$ , i.e., the Upper Half Plane (UHP), in the counter-clockwise direction,

$$\oint_{C:\text{UHP}} \frac{e^{az}}{1+e^z} dz, \quad \text{where } 0 < a < 1.$$

參考用

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