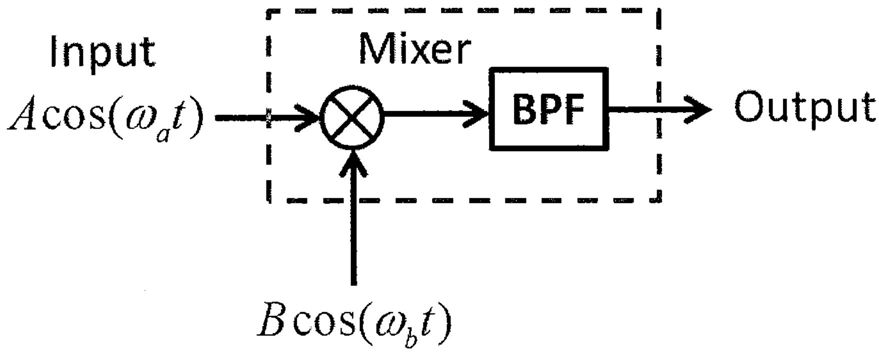
計算題，請寫出計算過程。

Note: Detailed derivations are required to obtain a full score for each problem.

1. (12%) A "mixer" is an RF electric circuit that is commonly used in communication systems to shift the EM wave's frequency to a desired frequency band by multiplying an input sinusoidal wave with another sinusoidal wave. The configuration of a mixer is simply a combination of an electric multiplier and a band-pass filter (BPF) as shown in the following diagram. Assume the input vector space is generated by n input sinusoidal waves with the same non-zero amplitude A, but different frequencies, $\omega_1, \omega_2, \cdots, \omega_n$, and the output vector space is generated by these n "down-converted" sinusoidal waves.
Note: "down-convert" means the frequency is reduced.

(a) (5%) Show that the input n signals are linearly independent.

(b) (7%) Please verify that if this mixer performs a linear transformation in this system.



2. (13%) Fourier series says that any periodic function can be expressed as a linear combination of infinite harmonic cosine and sine functions.

(a) (8%) If the period of a set of periodic functions is 2π, what is the orthonormal basis of the vector space that these periodic functions belong to? You have to show the vectors in the basis are orthonormal to get full credit.

(b) (5%) From the spectral theorem, find a vector $g(x)$ in a subspace spanned by only $\{1, \cos(x), \cos(2x), \sin(x), \sin(2x)\}$ so that $g(x)$ has the shortest distance to the function $f(x) = x$ in the interval $[-\pi, \pi]$.
Note: In this problem, you may need: $\int x \sin ax dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$ and $\int x \cos ax dx = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax$

3. (25%) Which of the following statements is TRUE and which one is FALSE? Justify your answers.

(a) (5%) The set $\mathcal{S} \triangleq \{\mathbf{A} \mid \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{A}^T = -\mathbf{A}\}$ is not a subspace.

(b) (5%) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Suppose the system $\mathbf{Ax} = \mathbf{0}_n$ has infinitely many solutions and $\mathbf{Bx} = \mathbf{0}_n$ has only one solution. This implies the system $\mathbf{ABx} = \mathbf{0}_n$ has exactly one solution.

(c) (5%) If $\mathbf{A}$ has an eigenvector $\mathbf{x}$ with eigenvalue λ, then $\exp(\mathbf{A})$ has $\mathbf{x}$ as an eigenvector with the eigenvalue $\exp(\lambda)$.

(d) (5%) If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{AB} - \mathbf{BA} = \mathbf{A}$, then $\mathbf{A}$ is singular.

(e) (5%) Consider the following linear equations characterized by $\mathbf{A} \in \mathbb{R}^{m\times n}$:

$$\mathbf{Ax} = \mathbf{b}.$$

Then $\widehat{\mathbf{x}} = \mathbf{A}^{\dagger}\mathbf{b} + \mathbf{v}$ is a solution to the above linear equations, where $\mathbf{A}^{\dagger}$ is the pseudo-inverse of $\mathbf{A}$ and $\mathbf{v} \in \mathcal{N}(\mathbf{A})$ (null space of $\mathbf{A}$). If your answer is "False", please describe the physical meaning of $\mathbf{A}\widehat{\mathbf{x}}$.

4. (16%) Consider a vector of d independent binary random variables $\mathbf{a}$=$[a_0, a_1, \cdots, a_{d-1}]$ in which $\Pr(a_\ell = 1)$=$p_1^{(\ell)}$, $\Pr(a_\ell = 0)$=$p_0^{(\ell)}$, and $L_\ell = \ln(\frac{p_0^{(\ell)}}{p_1^{(\ell)}})$ for $\ell = 0, 1, \cdots, d-1$.

(a) (6%) Consider a new binary random variable A which is the binary sum of a_1 and a_2, i.e., $A = a_1 \oplus a_2$. Please show that the log-likelihood ratio of A defined as $L(A) = \ln(\frac{\Pr(a=0)}{\Pr(a=1)})$, can be expressed as $L(A) = \ln(\frac{1+\exp(L_1+L_2)}{\exp(L_1)+\exp(L_2)})$.

(b) (10%) Please show that the probability that $\mathbf{a}$ contains an even number of 1s is $\frac{1}{2} + \frac{1}{2}\Pi_{\ell=0}^{d-1}(1 - 2p_1^{(\ell)})$.

5. (17%) Consider a Gaussian distributed random variable X with mean μ_X and variance σ_X^2. The moment-generating function of X is defined as $M_X(t) = E[e^{tX}]$.

(a) (3%) Derive the moment-generating function of X.

(b) (3%) If the first moment and second moment of X are 2 and 20, respectively, find the moment-generating function $M_Z(t)$ of the random variable $Z = \frac{(X+6)}{2}$ based on the result obtained from (a).

(c) (2%) Based on the result obtained from (b), find the mean and variance of Z.

(d) (4%) Find the correlation coefficient between X and Z^2.

(e) (2%) What can you say about the range of the probability of $Z \geq 10$?

(f) (3%) Assume that Alice try to estimate the mean of Z based on a random sample $Z_1, Z_2, \cdots, Z_N$ of size N taken from the distribution of Z. Find the minimum sample size required to ensure that the estimation error is smaller than 0.5 with a probability of 0.9.

6. (17%) Random variables X and Y have joint probability density function

$$f_{XY} = \begin{cases} \frac{5x^2}{2}, & -1 \le x \le 1, 0 \le y \le x^2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $A = \{Y \le \frac{1}{4}\}$.

(a) (4%) Find $f_X(x)$.

(b) (4%) Find $f_{XY|A}(x, y)$.

(c) (4%) Find $f_{Y|A}(y)$.

(d) (5%) Find $E[Y|A]$.